TERMINOLOGY

**Axis:** A line around which a curve is reflected e.g. the axis of symmetry of a parabola

**Cartesian equation:** An equation involving two variables \( x \) and \( y \)

**Chord:** An interval joining any two points on a curve. In this chapter, any two points on a parabola

**Circle:** The locus of a point moving so that it is equidistant from a fixed point on a plane surface

**Directrix:** A fixed line from which all points equidistant from this line and a fixed point called the focus form a parabola

**Focal chord:** A chord that passes through the focus

**Focal length:** The distance between the focus and the vertex of a parabola or the shortest distance between the vertex and the directrix

**Focus:** A fixed point from which all points equidistant from this point and the directrix form a parabola

**Latus rectum:** A focal chord that is perpendicular to the axis of the parabola

**Locus:** The path traced out by a point that moves according to a particular pattern or rule. Locus can be described algebraically or geometrically

**Parabola:** The locus of a point moving so that it is equidistant from a fixed point called the focus and a fixed line called the directrix

**Parametric equations:** A set of equations where variables \( x \) and \( y \) are both written in terms of a third variable, called a parameter, usually \( p \) or \( t \)

**Tangent:** A straight line that touches a curve at a single point only.

**Vertex:** The turning point (maximum or minimum point) of a parabola. It is the point where the parabola meets the axis of symmetry
INTRODUCTION

THIS CHAPTER EXPANDS THE work on functions that you have already learned. It shows a method of finding the equation of a locus. In particular, you will study the circle and the parabola, defined as a locus.

A parabola can also be defined as a set of parametric equations, and you will study these in this chapter.

DID YOU KNOW?

Locus problems have been studied since very early times. Apollonius of Perga (262–190 BC), a contemporary (and rival) of Archimedes, studied the locus of various figures. In his Plane Loci, he described the locus points whose ratio from two fixed points is constant. This locus is called the ‘Circle of Apollonius’.

Apollonius also used the equation $y^2 = 4ax$ for the parabola.

René Descartes (1596–1650) was another mathematician who tried to solve locus problems. His study of these led him to develop analytical (coordinate) geometry.

Locus

A relation can be described in two different ways. It can be a set of points that obey certain conditions, or a single point that moves along a path according to certain conditions.

A locus is the term used to describe the path of a single moving point that obeys certain conditions.
EXAMPLES

Describe the locus of the following.

1. A pencil on the end of compasses.

Solution

The path of the pencil is a circle with centre at the point of the compasses.

2. A person going up an escalator (standing still on one step).

Solution

The body travels along a straight line parallel to the escalator.

3. A doorknob on a closing door.
Solution

If the door could swing right around it would follow a circle. So a door closing swings through an arc of a circle.

4. A point on the number line that is 3 units from 0.

Solution

\[ \text{The locus is } \pm 3. \]

5. A point in the number plane that moves so that it is always 3 units from the y-axis.

Solution

\[ \text{The locus is 2 vertical lines with equations } x = \pm 3. \]

Class Discussion

Describe the path of a person abseiling down a cliff.

11.1 Exercises

Describe the locus of the following:

1. a racing car driving around a track
2. a person climbing a ladder
3. a child on a swing
4. a ball’s flight when thrown
5. a person driving up to the 5th floor of a car park
6. a point that moves along the number line such that it is always less than 2 units from 0

7. a point on the number plane that moves so that it is always 2 units from the origin

8. a point that moves so that it is always 1 unit from the x-axis

9. a point that moves so that it is always 5 units from the y-axis

10. a point that moves so that it is always 2 units above the x-axis

11. a point that moves so that it is always 1 unit from the origin

12. a point that moves so that it is always 4 units from the point (1, −2)

13. a point that is always 5 units below the x-axis

14. a point that is always 3 units away from the point (1, 1)

15. a point that is always 7 units to the left of the y-axis

16. a point that is always 3 units to the right of the y-axis

17. a point that is always 8 units from the x-axis

18. a point that is always 4 units from the y-axis

19. a point that is always 6 units from the point (−2, 4)

20. a point that is always 1 unit from the point (−4, 5).

A locus describes a single point $P(x, y)$ that moves along a certain path. The equation of a locus can often be found by using $P(x, y)$ together with the information given about the locus.

**EXAMPLES**

1. Find the equation of the locus of a point $P(x, y)$ that moves so that it is always 3 units from the origin.

**Solution**

You may recognise this locus as a circle, centre (0, 0) radius 3 units. Its equation is given by $x^2 + y^2 = 9$.

Alternatively, use the distance formula.

$\begin{align*}
   d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
   \text{or } d^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2
\end{align*}$
Let \( P(x, y) \) be a point of the locus.

We want \( PO = 3 \)

i.e. \( PO^2 = 9 \)

\[
(x - 0)^2 + (y - 0)^2 = 9
\]

\[
x^2 + y^2 = 9
\]

2. Find the equation of the locus of point \( P(x, y) \) that moves so that distance \( PA \) to distance \( PB \) is in the ratio 2:1 where \( A = (-3, 1) \) and \( B = (2, -2) \).

**Solution**

Let \( P(x, y) \) be a point of the locus.

\[
PA : PB = 2:1
\]

i.e.

\[
\frac{PA}{PB} = \frac{2}{1}
\]

\[
PA = 2PB
\]

\[
PA^2 = (2PB)^2
\]

i.e.

\[
[x - (-3)]^2 + (y - 1)^2 = 4\left\{ [x - 2]^2 + [y - (-2)]^2 \right\}
\]

\[
(x + 3)^2 + (y - 1)^2 = 4\left\{ [x - 2]^2 + (y + 2)^2 \right\}
\]

\[
x^2 + 6x + 9 + y^2 - 2y + 1 = 4(x^2 - 4x + 4 + y^2 + 4x + 4)
\]

\[
x^2 + 6x + 9 + y^2 - 2y + 1 = 4x^2 - 16x + 16 + 4y^2 + 16y + 16
\]

\[
0 = 3x^2 - 22x + 3y^2 + 18y + 22
\]

or

\[
3x^2 - 22x + 3y^2 + 18y + 22 = 0
\]

3. Find the equation of the locus of a point \( P(x, y) \) that moves so that the line \( PA \) is perpendicular to line \( PB \), where \( A = (1, 2) \) and \( B = (-3, -1) \).
Solution

Let \( P(x, y) \) be a point of the locus.
For perpendicular lines, \( m_1 m_2 = -1 \)

Using \( m = \frac{y_2 - y_1}{x_2 - x_1} \)

\[ PA: \quad m_1 = \frac{y - 2}{x - 1} \]
\[ PB: \quad m_2 = \frac{y - (-1)}{x - (-3)} = \frac{y + 1}{x + 3} \]

For \( PA \) perpendicular to \( PB \)

\[ \frac{y - 2}{x - 1} \times \frac{y + 1}{x + 3} = -1 \]
\[ \frac{y^2 - y - 2}{x^2 + 2x - 3} = -1 \]
\[ y^2 - y - 2 = -(x^2 + 2x - 3) \]
\[ = -x^2 - 2x + 3 \]

i.e. \( x^2 + 2x + y^2 - y - 5 = 0 \)

4. Find the equation of the locus of point \( P(x, y) \) that is equidistant from fixed point \( A(1, -2) \) and fixed line with equation \( y = 5 \).

Solution

Let \( P(x, y) \) be a point of the locus.
\( B \) has coordinates \( (x, 5) \).
We want \( PA = PB \)
i.e. \( PA^2 = PB^2 \)

\[ (x - 1)^2 + (y - (-2))^2 = (x - x)^2 + (y - 5)^2 \]
\[ (x - 1)^2 + (y + 2)^2 = (y - 5)^2 \]
\[ x^2 - 2x + 1 + y^2 + 4y + 4 = y^2 - 10y + 25 \]
\[ x^2 - 2x + 14y - 20 = 0 \]
1. Find the equation of the locus of point \( P(x, y) \) that moves so that it is always 1 unit from the origin.

2. Find the equation of the locus of point \( P(x, y) \) that moves so that it is always 9 units from the point \((-1, -1)\).

3. Find the equation of the locus of a point that moves so that it is always 2 units from the point \((5, -2)\).

4. Find the equation of the locus of point \( P(x, y) \) that moves so that it is equidistant from the points \((3, 2)\) and \((-1, 5)\).

5. Find the equation of the locus of a point that moves so that it is equidistant from the points \((-4, 6)\) and \((2, -7)\).

6. Find the equation of the locus of point \( P(x, y) \) that moves so that it is equidistant from the \(x\)-axis and the \(y\)-axis.

7. Find the equation of the locus of a point \( P \) that moves so that \( PA \) is twice the distance of \( PB \) where \( A = (0, 3) \) and \( B = (4, 7) \).

8. Find the equation of the locus of point \( P(x, y) \) that moves so that the ratio of \( PA \) to \( PB \) is 3:2 where \( A = (-6, 5) \) and \( B = (3, -1) \).

9. Find the equation of the locus of a point that moves so that it is equidistant from the point \((2, -3)\) and the line \( y = 7 \).

10. Find the equation of the locus of a point that moves so that it is equidistant from the point \((0, 5)\) and the line \( y = -5 \).

11. Find the equation of the locus of a point that moves so that it is equidistant from the point \((2, 0)\) and the line \( x = 6 \).

12. Find the equation of the locus of a point that moves so that it is equidistant from the point \((1, -1)\) and the line \( y = 3 \).

13. Find the equation of the locus of a point that moves so that it is equidistant from the point \((0, -3)\) and the line \( y = 3 \).

14. Find the equation of the locus of a point \( P(x, y) \) that moves so that the line \( PA \) is perpendicular to line \( PB \) where \( A = (1, -3) \) and \( B = (4, 5) \).

15. Find the equation of the locus of a point \( P(x, y) \) that moves so that the line \( PA \) is perpendicular to line \( PB \), where \( A = (-4, 0) \) and \( B = (1, 1) \).

16. Find the equation of the locus of a point \( P(x, y) \) that moves so that the line \( PA \) is perpendicular to line \( PB \) where \( A = (1, 5) \) and \( B = (-2, -3) \).

17. Point \( P \) moves so that \( PA^2 + PB^2 = 4 \) where \( A = (3, -1) \) and \( B = (-5, 4) \). Find the equation of the locus of \( P \).

18. Point \( P \) moves so that \( PA^2 + PB^2 = 12 \) where \( A = (-2, -5) \) and \( B = (1, 3) \). Find the equation of the locus of \( P \).

19. Find the equation of the locus of a point that moves so that its distance from the line \( 3x + 4y + 5 = 0 \) is always 4 units.
20. Find the equation of the locus of a point that moves so that its distance from the line $12x - 5y - 1 = 0$ is always 1 unit.

21. Find the equation, in exact form, of the locus of a point that moves so that its distance from the line $x - 2y - 3 = 0$ is always 5 units.

22. Find the equation of the locus of a point that moves so that it is equidistant from the line $4x - 3y + 2 = 0$ and the line $3x + 4y - 7 = 0$.

23. Find the equation of the locus of a point that moves so that it is equidistant from the line $3x + 4y - 5 = 0$ and the line $5x + 12y - 1 = 0$.

24. Given two points $A(3, -2)$ and $B(-1, 7)$, find the equation of the locus of $P(x, y)$ if the gradient of $PA$ is twice the gradient of $PB$.

25. If $R$ is the fixed point $(3, 2)$ and $P$ is a movable point $(x, y)$, find the equation of the locus of $P$ if the distance $PR$ is twice the distance from $P$ to the line $y = -1$.

**PROBLEM**

*Can you see 2 mistakes in the solution to this question?*

Find the locus of point $P(x, y)$ that moves so that its perpendicular distance from the line $12x + 5y - 1 = 0$ is always 3 units.

**Solution**

Let $P(x, y)$ be a point of the locus.

\[
d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}
\]

\[
3 = \frac{|5x + 12y - 1|}{\sqrt{5^2 + 12^2}}
\]

\[
= \frac{|5x + 12y - 1|}{\sqrt{25 + 144}}
\]

\[
= \frac{|5x + 12y - 1|}{\sqrt{169}}
\]

\[
= \frac{|5x + 12y - 1|}{13}
\]

\[
\therefore 39 = 5x + 12y - 1
\]

\[
0 = 5x + 12y - 40
\]

*Can you find the correct locus?*
Circle as a Locus

The locus of point $P(x, y)$ that is always a constant distance from a fixed point is a circle.

The circle, centre $(0, 0)$ and radius $r$, has the equation

$$x^2 + y^2 = r^2$$

Proof

Find the equation of the locus of point $P(x, y)$ that is always $r$ units from the origin.

Let $P(x, y)$ be a point of the locus.  

$$OP = r$$  

i.e.  

$$OP^2 = r^2$$  

$$(x - 0)^2 + (y - 0)^2 = r^2$$

$$x^2 + y^2 = r^2$$

So $x^2 + y^2 = r^2$ is the equation of the locus. It describes a circle with radius $r$ and centre $(0, 0)$.

The circle, centre $(a, b)$ and radius $r$, has the equation

$$(x - a)^2 + (y - b)^2 = r^2$$

Proof

Find the equation of the locus of point $P(x, y)$ that is always $r$ units from point $A(a, b)$.  

Let $P(x, y)$ be a point of the locus.

\[ AP = r \]

i.e.
\[ AP^2 = r^2 \]
\[ (x - a)^2 + (y - b)^2 = r^2 \]

So $(x - a)^2 + (y - b)^2 = r^2$ is the equation of the locus. It describes a circle with radius $r$ and centre $(a, b)$.

**EXAMPLES**

1. Find the equation of the locus of a point that is always 2 units from the point $(-1, 0)$.

**Solution**

This is a circle with radius 2 and centre $(-1, 0)$. Its equation is in the form
\[ (x - a)^2 + (y - b)^2 = r^2 \]

i.e. $[x - (-1)]^2 + (y - 0)^2 = 2^2$
\[ (x + 1)^2 + y^2 = 4 \]
\[ x^2 + 2x + 1 + y^2 = 4 \]
\[ x^2 + 2x + y^2 - 3 = 0 \]

2. Find the radius and the coordinates of the centre of the circle $x^2 + 2x + y^2 - 6y - 15 = 0$.

**Solution**

We put the equation into the form $(x - a)^2 + (y - b)^2 = r^2$.
To do this we complete the square.
In general, to complete the square on $x^2 + bx$, add $\left( \frac{b}{2} \right)^2$ to give:
\[ x^2 + bx + \left( \frac{b}{2} \right)^2 = \left( x + \frac{b}{2} \right)^2 \]
First we move any constants to the other side of the equation, then complete the square.
To complete the square on $x^2 + 2x$, we add $\left( \frac{2}{2} \right)^2 = 1$. 
To complete the square on \( y^2 - 6y \), we add \( \left( \frac{6}{2} \right)^2 = 9 \).

\[
x^2 + 2x + y^2 - 6y - 15 = 0
\]
\[
x^2 + 2x + y^2 - 6y = 15
\]
\[
x^2 + 2x + 1 + y^2 - 6y + 9 = 15 + 1 + 9
\]
\[
(x + 1)^2 + (y - 3)^2 = 25
\]
\[
(x - (-1))^2 + (y - 3)^2 = 5^2
\]
The equation is in the form \((x - a)^2 + (y - b)^2 = r^2\).
This is a circle, centre \((-1, 3)\) and radius 5.

### 11.3 Exercises

1. Find the length of the radius and the coordinates of the centre of each circle.
   (a) \( x^2 + y^2 = 100 \)
   (b) \( x^2 + y^2 = 5 \)
   (c) \( (x - 4)^2 + (y - 5)^2 = 16 \)
   (d) \( (x - 5)^2 + (y + 6)^2 = 49 \)
   (e) \( x^2 + (y - 3)^2 = 81 \)

2. Find the equation of each circle in expanded form (without grouping symbols).
   (a) Centre \((0, 0)\) and radius 4
   (b) Centre \((3, 2)\) and radius 5
   (c) Centre \((-1, 5)\) and radius 3
   (d) Centre \((2, 3)\) and radius 6
   (e) Centre \((-4, 2)\) and radius 5
   (f) Centre \((0, -2)\) and radius 1
   (g) Centre \((4, 2)\) and radius 7
   (h) Centre \((-3, -4)\) and radius 9
   (i) Centre \((-2, 0)\) and radius \(\sqrt{5}\)
   (j) Centre \((-4, -7)\) and radius \(\sqrt{3}\).

3. Find the equation of the locus of a point moving so that it is 1 unit from the point \((9, -4)\).

4. Find the equation of the locus of a point moving so that it is 4 units from the point \((-2, -2)\).

5. Find the equation of the locus of a point moving so that it is 7 units from the point \((1, 0)\).

6. Find the equation of the locus of a point moving so that it is 2 units from the point \((-3, 8)\).

7. Find the equation of the locus of a point moving so that it is \(\sqrt{2}\) units from the point \((5, -2)\).

8. Find the equation of a circle with centre \((0, 0)\) and radius 3 units.

9. Find the equation of a circle with centre \((1, 5)\) and radius 1 unit.

10. Find the equation of a circle with centre \((-6, 1)\) and radius 6 units.

11. Find the equation of a circle with centre \((4, 3)\) and radius \(\sqrt{3}\) units.

12. Find the equation of a circle with centre \((0, -3)\) and radius \(2\sqrt{2}\) units.

13. Find the coordinates of the centre and the length of the radius of each circle.
   (a) \( x^2 - 4x + y^2 - 2y - 4 = 0 \)
   (b) \( x^2 + 8x + y^2 - 4y - 5 = 0 \)
   (c) \( x^2 + y^2 - 2y = 0 \)
(d) \(x^2 - 10x + y^2 + 6y - 2 = 0\)
(e) \(x^2 + 2x + y^2 - 2y + 1 = 0\)
(f) \(x^2 - 12x + y^2 = 0\)
(g) \(x^2 + 6x + y^2 - 8y = 0\)
(h) \(x^2 + 20x + y^2 - 4y + 40 = 0\)
(i) \(x^2 - 14x + y^2 + 2y + 25 = 0\)
(j) \(x^2 + 2x + y^2 + 4y - 5 = 0\)

14. Find the centre and radius of the circle with equation given by \(x^2 - 6x + y^2 + 2y - 6 = 0\).

15. Find the centre and radius of the circle with equation given by \(x^2 - 4x + y^2 - 10y + 4 = 0\).

16. Find the centre and radius of the circle with equation given by \(x^2 + 2x + y^2 + 12y - 12 = 0\).

17. Find the centre and radius of the circle with equation given by \(x^2 - 8x + y^2 - 14y + 1 = 0\).

18. Find the centre and radius of the circle with equation given by \(x^2 + 3x + y^2 - 2y - 3 = 0\).

19. Sketch the circle whose equation is given by \(x^2 + 4x + y^2 - 2y + 1 = 0\).

20. Prove that the line \(3x + 4y + 21 = 0\) is a tangent to the circle \(x^2 - 8x + y^2 + 4y - 5 = 0\).

21. (a) Show that \(x^2 - 2x + y^2 + 4y + 1 = 0\) and \(x^2 - 2x + y^2 + 4y - 4 = 0\) are concentric.
   (b) Find the difference between their radii.

22. Given two points \(A(2, -5)\) and \(B(-4, 3)\), find the equation of the circle with diameter \(AB\).

23. Find the exact length of the tangent from \((4, -5)\) to the circle \(x^2 + 4x + y^2 - 2y - 11 = 0\).

24. Find the exact length of \(AB\) where \(A\) and \(B\) are the centres of the circles \(x^2 - 6x + y^2 = 0\) and \(x^2 + 4x + y^2 + 6y - 3 = 0\) respectively.

25. (a) Find the length of \(XY\) where \(X\) and \(Y\) are the centres of the circles \(x^2 + 6x + y^2 - 2y + 1 = 0\) and \(x^2 - 4x + y^2 - 2y + 1 = 0\) respectively.
   (b) Find the radius of each circle.
   (c) What conclusion can you draw from the results for (a) and (b)?

26. Show that the circles \(x^2 + y^2 = 4\) and \(x^2 + 2x + y^2 - 4y - 4 = 0\) both have \(3x + 4y + 10 = 0\) as a tangent.

27. A circle has centre \(C(-1, 3)\) and radius 5 units.
   (a) Find the equation of the circle.
   (b) The line \(3x - y + 1 = 0\) meets the circle at two points. Find their coordinates.
   (c) Let the coordinates be \(X\) and \(Y\), where \(Y\) is the coordinate directly below the centre \(C\). Find the coordinates of point \(Z\), where \(YZ\) is a diameter of the circle.
   (d) Hence show \(\angle ZXY = 90^\circ\).

28. (a) Find the perpendicular distance from \(P(2, -5)\) to the line \(5x + 12y - 2 = 0\).
   (b) Hence find the equation of the circle with centre \(P\) and tangent \(5x + 12y - 2 = 0\).

Concentric circles have the same centre.
Parabola as a Locus

The locus of a point that is **equidistant from a fixed point and a fixed line** is always a parabola. The **fixed point** is called the **focus** and the **fixed line** is called the **directrix**.

![Diagram of a parabola with focus and directrix](image)

Work on the parabola as a locus is very important, as the properties of the parabola are useful to us. The parabola is used in lenses of glasses and cameras, in car headlights, and for bridges and radio telescope dishes.

**DID YOU KNOW?**

Any rope or chain **supporting a load** (e.g. a suspension bridge) is in the shape of a **parabola**.

Find some examples of suspension bridges that have a parabola shaped chain.

Other bridges have ropes or chains hanging freely. These are not in the shape of a parabola, but are in a shape called a catenary. Can you find some bridges with this shape?

More recent bridges are cable-stayed, where ropes or chains are attached to towers, or pylons, and fan out along the sides of the bridge. An example is the Anzac Bridge in Sydney.

There are many different bridge designs. One famous bridge in Australia is the Sydney Harbour Bridge.

Research different bridge designs and see if you can find some with parabolic shapes.

Parabola with vertex at the origin

Just as the circle has a special equation when its centre is at the origin, the parabola has a special equation when its vertex is at the origin. Both also have a more general formula.
The locus of a point that is equidistant from a fixed point and a fixed line is always in the shape of a parabola.

If the fixed point is \((0, a)\) and the fixed line is \(y = -a\) (where \(a > 0\)), then one of the equidistant points is the origin \((0, 0)\). The distance between the points \((0, 0)\) and \((0, a)\) is \(a\) units.

The point on \(y = -a\) directly below the origin is \((0, -a)\) and the distance from \((0, 0)\) to \((0, -a)\) is also \(a\) units.

To find the equation of the parabola, we use the general process to find the equation of any locus. The features of the parabola have special names.

A parabola is equidistant from a fixed point and a fixed line.
- The fixed point is called the focus.
- The fixed line is called the directrix.
- The turning point of the parabola is called the vertex.
- The axis of symmetry of the parabola is called its axis.
- The distance between the vertex and the focus is called the focal length.
- An interval joining any two points on the parabola is called a chord.
- A chord that passes through the focus is called a focal chord.
- The focal chord that is perpendicular to the axis is called the latus rectum.
- A tangent is a straight line that touches the parabola at a single point.
PARABOLA \( x^2 = 4ay \)

The locus of point \( P(x, y) \) moving so that it is equidistant from the point \((0, a)\) and the line \( y = -a \) is a parabola with equation \( x^2 = 4ay \)

**Proof**

![Diagram of a parabola with labeled points and distances]

Let \( P(x, y) \) be a point of the locus.

Taking the perpendicular distance from \( P \) to the line \( y = -a \), point \( B(x, -a) \).

\[
PA = PB \\
\therefore PA^2 = PB^2 \\
(x - 0)^2 + (y - a)^2 = (x - x)^2 + [y - (-a)]^2 \\
x^2 + (y - a)^2 = (y + a)^2 \\
x^2 + y^2 - 2ay + a^2 = y^2 + 2ay + a^2 \\
x^2 = 4ay
\]

The parabola \( x^2 = 4ay \) has
- **focus** at \((0, a)\)
- **directrix** with equation \( y = -a \)
- **vertex** at \((0, 0)\)
- **axis** with equation \( x = 0 \)
- **focal length** the distance from the vertex to the focus with length \( a \)
- **latus rectum** that is a horizontal focal chord with length \( 4a \)

**Since the focal length is \( a \), \( a \) is always a positive number**

**Class Investigation**

Find the equation of the locus if point \( P(x, y) \) is equidistant from \((0, -a)\) and \( y = a \).
EXAMPLES

1. Find the equation of the parabola whose focus has coordinates (0, 2) and whose directrix has equation \( y = -2 \).

**Solution**

The focus has coordinates in the form \((0, a)\) and the directrix has equation in the form \( y = -a \), where \( a = 2 \).

\[ \therefore \text{the parabola is in the form } x^2 = 4ay \text{ where } a = 2 \]

i.e. \( x^2 = 4(2)y \)

\[ x^2 = 8y \]

2.

(a) Find the coordinates of the focus and the equation of the directrix of the parabola \( x^2 = 20y \).

(b) Find the points on the parabola at the endpoints of the latus rectum and find its length.

**Solution**

(a) The parabola \( x^2 = 20y \) is in the form \( x^2 = 4ay \)

\[ 4a = 20 \]

\[ \therefore a = 5 \]

The focal length is 5 units.

We can find the coordinates of the focus and the equation of the directrix in two ways.

Method 1:
Draw the graph \( x^2 = 20y \) and count 5 units up and down from the origin as shown.

The focus is \((0, 5)\) and the directrix has equation \( y = -5 \).
Method 2:
The focus is in the form \((0, a)\) where \(a = 5\).
So the focus is \((0, 5)\).

The directrix is in the form \(y = -a\) where \(a = 5\).
So the directrix is \(y = -5\).

(b) The latus rectum is a focal chord that is perpendicular to the axis of the parabola as shown

![Graph of a parabola with focus at (0, 5) and directrix at y = -5. The latus rectum is shown as a focal chord perpendicular to the axis of the parabola.]

The endpoints of the latus rectum will be where the line \(y = 5\) and the parabola intersect.
Substitute \(y = 5\) into the parabola.

\[
x^2 = 20y
\]

\[
x = 20(5)
\]

\[
x = 100
\]

\[
x = \pm \sqrt{100}
\]

\[
x = \pm 10
\]

So the endpoints are \((-10, 5)\) and \((10, 5)\).

![Graph showing the parabola with the latus rectum between \((-10, 5)\) and \((10, 5)\).]

From the graph, the length of the latus rectum is 20 units.
3. Find the equation of the focal chord to the parabola \( x^2 = 4y \) that passes through \((-4, 4)\).

**Solution**

The parabola \( x^2 = 4y \) is in the form \( x^2 = 4ay \).

\[
4a = 4
\]

\[
\therefore \quad a = 1
\]

The focal length is 1 unit.

The focus is 1 unit up from the origin at \((0, 1)\) and the focal chord also passes through \((-4, 4)\).

We can find the equation of the line between \((0, 1)\) and \((-4, 4)\) by using either formula

\[
\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}
\]

\[
\frac{y - 1}{x} = \frac{4 - 1}{-4 - 0}
\]

\[
\frac{y - 1}{x} = \frac{3}{-4}
\]

\[
-4(y - 1) = 3x
\]

\[
-4y + 4 = 3x
\]

\[
0 = 3x + 4y - 4
\]

As you saw in the previous chapter, a parabola can be concave downwards. Can you guess what the equation of this parabola might be?

**PARABOLA** \( x^2 = -4ay \)

The locus of a point \( P(x, y) \) moving so that it is equidistant from the point \((0, -a)\) and the line \( y = a \) is a parabola with equation \( x^2 = -4ay \).
**Proof**

Let $P(x, y)$ be a point of the locus.

Taking the perpendicular distance from $P$ to the line $y = a$, point $B = (x, a)$.

\[ PA = PB \]

\[ PA^2 = PB^2 \]

\[
(x - 0)^2 + [y - (-a)]^2 = (x - x)^2 + (y - a)^2 \\
x^2 + (y + a)^2 = (y - a)^2 \\
x^2 + y^2 + 2ay + a^2 = y^2 - 2ay + a^2 \\
x^2 = -4ay
\]

The parabola $x^2 = -4ay$ has
- **focus** at $(0, -a)$
- **directrix** with equation $y = a$
- **vertex** at $(0, 0)$
- **axis** with equation $x = 0$
- **focal length** $a$
- **latus rectum** a horizontal focal chord with length $4a$
EXAMPLES

1. Find the equation of the parabola with focus \((0, -4)\) and directrix \(y = 4\).

Solution

If we draw this information, the focus is below the directrix as shown. So the parabola will be concave downwards (the parabola always turns away from the directrix).

The focal length is 4 so \(a = 4\).

The parabola is in the form \(x^2 = -4ay\) where \(a = 4\).

\[
\begin{align*}
x^2 &= -4ay \\
&= -4(4)y \\
&= -16y
\end{align*}
\]

2. Find the coordinates of the vertex, the coordinates of the focus and the equation of the directrix of the parabola \(x^2 = -12y\).

Solution

The parabola \(x^2 = -12y\) is in the form \(x^2 = -4ay\).

\[
4a = 12
\]

\[
\therefore a = 3
\]

The focal length is 3 units.

The vertex is \((0, 0)\).

We can find the coordinates of the focus and the equation of the directrix in two ways.

Method 1:

Draw the graph \(x^2 = -12y\) and count 3 units up and down from the origin as shown. (The parabola is concave downward.)
Counting down 3 units, the focus is $(0, -3)$.
Counting up 3 units, the directrix has equation $y = 3$.

Method 2:
The focus is in the form $(0, -a)$ where $a = 3$.
So the focus is $(0, -3)$.
The directrix is in the form $y = a$ where $a = 3$.
So the directrix is $y = 3$.

3. Find the equation of the parabola with focal length 5 and whose vertex is $(0, 0)$ and equation of the axis is $x = 0$.

**Solution**

Vertex $(0, 0)$ and axis given by $x = 0$ give a parabola in the form $x^2 = \pm 4ay$, since there is not enough information to tell whether it is concave upwards or downwards.
This gives two possible parabolas.
Focal length of 5 means $a = 5$

The equation is $x^2 = \pm 4(5)y$

i.e. $x^2 = \pm 20y$

### 11.4 Exercises

1. Find the equation of each parabola.
   - (a) focus $(0, 5)$, directrix $y = -5$
   - (b) focus $(0, 9)$, directrix $y = -9$
   - (c) focus $(0, 1)$, directrix $y = -1$
   - (d) focus $(0, 4)$, directrix $y = -4$
   - (e) focus $(0, 10)$, directrix $y = -10$
   - (f) focus $(0, 3)$, directrix $y = -3$
   - (g) focus $(0, 6)$, directrix $y = -6$
   - (h) focus $(0, 11)$, directrix $y = -11$
   - (i) focus $(0, 2)$, directrix $y = -2$
   - (j) focus $(0, 12)$, directrix $y = -12$

2. Find the equation of each parabola.
   - (a) focus $(0, -1)$, directrix $y = 1$
   - (b) focus $(0, -3)$, directrix $y = 3$
   - (c) focus $(0, -4)$, directrix $y = 4$
   - (d) focus $(0, -7)$, directrix $y = 7$
   - (e) focus $(0, -6)$, directrix $y = 6$
   - (f) focus $(0, -9)$, directrix $y = 9$
   - (g) focus $(0, -8)$, directrix $y = 8$
   - (h) focus $(0, -2)$, directrix $y = 2$
   - (i) focus $(0, -15)$, directrix $y = 15$
   - (j) focus $(0, -13)$, directrix $y = 13$

3. Find
   - (i) the coordinates of the focus and
   - (ii) the equation of the directrix of
     - (a) $x^2 = 4y$
     - (b) $x^2 = 28y$
     - (c) $x^2 = 16y$
     - (d) $x^2 = 36y$
     - (e) $x^2 = 40y$
     - (f) $x^2 = 44y$
     - (g) $x^2 = 12y$
     - (h) $x^2 = 6y$
     - (i) $x^2 = 10y$
     - (j) $x^2 = 15y$

4. Find
   - (i) the coordinates of the focus and
   - (ii) the equation of the directrix of
     - (a) $x^2 = -4y$
     - (b) $x^2 = -24y$
     - (c) $x^2 = -8y$
     - (d) $x^2 = -48y$
     - (e) $x^2 = -20y$
     - (f) $x^2 = -16y$
     - (g) $x^2 = -32y$
     - (h) $x^2 = -40y$
     - (i) $x^2 = -2y$
     - (j) $x^2 = -22y$

5. Find the equation of the parabola with
   - (a) coordinates of the focus $(0, 7)$ and equation of the directrix $y = -7$
   - (b) coordinates of the focus $(0, 11)$ and equation of the directrix $y = -11$
   - (c) coordinates of the focus $(0, -6)$ and equation of the directrix $y = 6$
   - (d) coordinates of the focus $(0, 2)$ and coordinates of the vertex $(0, 0)$. 


(e) coordinates of the vertex
(0, 0), equation of the axis \( x = 0 \) and focal length 3
(f) coordinates of the vertex
(0, 0), equation of the axis \( x = 0 \) and focal length 8
(g) coordinates of the vertex
(0, 0) and equation of the axis \( x = 0 \), and passing through the point \((-8, 2)\)
(h) coordinates of the vertex
(0, 0) and equation of the axis \( x = 0 \), and passing through the point \((-1, 7)\).

6. Find the coordinates of the focus, the equation of the directrix and the focal length of the parabola
(a) \( x^2 = 8y \)
(b) \( x^2 = 24y \)
(c) \( x^2 = -12y \)
(d) \( x^2 = 2y \)
(e) \( x^2 = -7y \)
(f) \( 2x^2 = y \)

7. Find the equation of the focal chord that cuts the curve \( x^2 = 8y \) at \((-4, 2)\).

8. The tangent with equation \( 2x - y - 4 = 0 \) touches the parabola \( x^2 = 4y \) at \(A\). Find the coordinates of \(A\).

9. The focal chord that cuts the parabola \( x^2 = -6y \) at \((6, -6)\) cuts the parabola again at \(X\). Find the coordinates of \(X\).

10. Find the coordinates of the endpoints of the latus rectum of the parabola \( x^2 = -8y \). What is the length of the latus rectum?

11. The equation of the latus rectum of a parabola is given by \( y = -3 \).
The axis of the parabola is \( x = 0 \), and its vertex is \((0, 0)\).
(a) Find the equation of the parabola.
(b) Find the equation of the directrix.
(c) Find the length of the focal chord that meets the parabola at \(\left(2, -\frac{1}{3}\right)\).

12. (a) Show that the point \((-3, 3)\) lies on the parabola with equation \( x^2 = 3y \).
(b) Find the equation of the line passing through \(P\) and the focus \(F\) of the parabola.
(c) Find the coordinates of the point \(R\) where the line \(PF\) meets the directrix.

13. (a) Find the equation of chord \(PQ\) where \(P\left(-1, \frac{1}{4}\right)\) and \(Q(2, 1)\) lie on the parabola \(x^2 = 4y\).
(b) Show that \(PQ\) is not a focal chord.
(c) Find the equation of the circle with centre \(Q\) and radius 2 units.
(d) Show that this circle passes through the focus of the parabola.

14. (a) Show that \(Q(2aq, aq^2)\) lies on the parabola \(x^2 = 4ay\).
(b) Find the equation of the focal chord through \(Q\).
(c) Prove that the length of the latus rectum is \(4a\).
Investigation

Sketch the parabola $x = y^2$. You may like to complete the table below to help you with its sketch.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$-3$</td>
</tr>
</tbody>
</table>

Is this parabola a function? What is its axis of symmetry?

The parabola that has $y^2$ rather than $x^2$ in its equation is a sideways parabola. It still has the same properties, but generally the $x$ and $y$ values are swapped around.

**PARABOLA** $y^2 = 4ax$

The locus of point $P(x, y)$ moving so that it is equidistant from the point $(a, 0)$ and the line $x = -a$ is a parabola with equation

$y^2 = 4ax$

**Proof**

Find the equation of the locus of point $P(x, y)$, which moves so that it is equidistant from the point $(a, 0)$ and the line $x = -a$.

Coordinates of $B$ are $(-a, y)$.

We want $PA = PB$

i.e. $PA^2 = PB^2$

$(x - a)^2 + (y - 0)^2 = [x - (-a)]^2 + (y - y)^2$

$(x - a)^2 + y^2 = (x + a)^2$

$x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2$

$y^2 = 4ax$
Chapter 11 Locus and the Parabola

The parabola $y^2 = 4ax$ has

- **focus** at $(a, 0)$
- equation of **directrix** $x = -a$
- **vertex** at $(0, 0)$
- **axis** with equation $y = 0$
- **focal length** the distance from the vertex to the focus with length $a$
- **latus rectum** that is a vertical focal chord with length $4a$

**EXAMPLES**

1. Find the equation of the parabola with focus $(7, 0)$ and directrix $x = -7$.

**Solution**

If we draw this information, the focus is to the right of the directrix as shown (the parabola always turns away from the directrix). So the parabola turns to the right.
The focal length is 7 so \( a = 7 \).
The parabola is in the form \( y^2 = 4ax \) where \( a = 7 \).
\[
y^2 = 4ax \\
= 4(7)x \\
= 28x.
\]

2. Find the coordinates of the focus and the equation of the directrix of the parabola \( y^2 = 32x \).

Solution

The parabola \( y^2 = 32x \) is in the form \( y^2 = 4ax \).
\[
4a = 32 \\
\therefore a = 8
\]
The focal length is 8 units.
Method 1:
Draw the graph \( y^2 = 32x \) and count 8 units to the left and right from the origin as shown. (The parabola turns to the right.)

Counting 8 units to the right, the focus is \((8, 0)\).
Counting 8 units to the left, the directrix has equation \( x = -8 \).

Method 2:
The focus is in the form \((a, 0)\) where \( a = 8 \).
So the focus is \((8, 0)\).
The directrix is in the form \( x = -a \) where \( a = 8 \).
So the directrix is \( x = -8 \).

A parabola can also turn to the left.
PARABOLA \( y^2 = -4ax \)

The locus of a point \( P(x, y) \) moving so that it is equidistant from the point \((-a, 0)\) and the line \( x = a \) is a parabola with equation \( y^2 = -4ax \)

**Proof**

Let \( P(x, y) \) be a point of the locus. 
Taking the perpendicular distance from \( P \) to the line \( x = a \), point \( B = (a, y) \).

\[
PA = PB
\]

\[
\therefore \quad PA^2 = PB^2
\]

\[
[x - (-a)]^2 + (y - 0)^2 = (x - a)^2 + (y - y)^2
\]

\[
(x + a)^2 + y^2 = (x - a)^2
\]

\[
x^2 + 2ax + a^2 + y^2 = x^2 - 2ax + a^2
\]

\[
y^2 = -4ax
\]
The parabola \( y^2 = -4ax \) has
- focus at \((-a, 0)\)
- directrix with equation \( x = a \)
- vertex at (0, 0)
- axis with equation \( y = 0 \)
- focal length \( a \)
- latus rectum a vertical focal chord with length \( 4a \)

**EXAMPLES**

1. Find the equation of the parabola with focus \((-4, 0)\) and directrix \( x = 4 \).

   **Solution**

   Drawing this information shows that the parabola turns to the left.

   The focal length is 4 so \( a = 4 \).
   The parabola is in the form \( y^2 = -4ax \) where \( a = 4 \).
   
   \[
   y^2 = -4ax \\
   = -4(4)x \\
   = -16x.
   \]

2. Find the coordinates of the focus and the equation of the directrix of the parabola \( y^2 = -2x \).

   **Solution**

   The parabola \( y^2 = -2x \) is in the form \( y^2 = -4ax \).
   
   \[
   4a = 2 \\
   \therefore a = \frac{1}{2}
   \]
   The focal length is \( \frac{1}{2} \) unit.
Method 1:
Draw the graph \( y^2 = -2x \) and count \( \frac{1}{2} \) unit to the left and right from the origin as shown. (The parabola turns to the left.)

Counting \( \frac{1}{2} \) units to the left, the focus is \( \left(-\frac{1}{2}, 0\right) \).

Counting \( \frac{1}{2} \) units to the right, the directrix has equation \( x = \frac{1}{2} \).

Method 2:
The focus is in the form \((-a, 0)\) where \( a = \frac{1}{2} \).
So the focus is \( \left(-\frac{1}{2}, 0\right) \).
The directrix is in the form \( xa = \) where \( a = \frac{1}{2} \).
So the directrix is \( x = \frac{1}{2} \).

11.5 Exercises

1. Find the equation of each parabola.
(a) focus \((2, 0)\), directrix \(x = -2\)
(b) focus \((5, 0)\), directrix \(x = -5\)
(c) focus \((14, 0)\), directrix \(x = -14\)
(d) focus \((9, 0)\), directrix \(x = -9\)
(e) focus \((8, 0)\), directrix \(x = -8\)
(f) focus \((6, 0)\), directrix \(x = -6\)
(g) focus \((7, 0)\), directrix \(x = -7\)
(h) focus \((3, 0)\), directrix \(x = -3\)
(i) focus \((4, 0)\), directrix \(x = -4\)
(j) focus \((1, 0)\), directrix \(x = -1\)

2. Find the equation of each parabola.
(a) focus \((-9, 0)\), directrix \(x = 9\)
(b) focus \((-4, 0)\), directrix \(x = 4\)
(c) focus \((-10, 0)\), directrix \(x = 10\)
(d) focus \((-6, 0)\), directrix \(x = 6\)
(e) focus \((-2, 0)\), directrix \(x = 2\)
(f) focus \((-12, 0)\), directrix \(x = 12\)
(g) focus \((-11, 0)\), directrix \(x = 11\)
(h) focus \((-5, 0)\), directrix \(x = 5\)
(i) focus \((-3, 0)\), directrix \(x = 3\)
(j) focus \((-7, 0)\), directrix \(x = 7\)
3. Find
(i) the coordinates of the focus and
(ii) the equation of the directrix of
(a) \( y^2 = 8x \)
(b) \( y^2 = 12x \)
(c) \( y^2 = 16x \)
(d) \( y^2 = 4x \)
(e) \( y^2 = 28x \)
(f) \( y^2 = 32x \)
(g) \( y^2 = 24x \)
(h) \( y^2 = 36x \)
(i) \( y^2 = x \)
(j) \( y^2 = 18x \)

4. Find
(i) the coordinates of the focus and
(ii) the equation of the directrix of
(a) \( y^2 = -8x \)
(b) \( y^2 = -12x \)
(c) \( y^2 = -28x \)
(d) \( y^2 = -4x \)
(e) \( y^2 = -24x \)
(f) \( y^2 = -52x \)
(g) \( y^2 = -60x \)
(h) \( y^2 = -2x \)
(i) \( y^2 = -26x \)
(j) \( y^2 = -5x \)

5. Find the equation of the parabola with
(a) coordinates of the focus \((5, 0)\) and equation of the directrix \(x = -5\)
(b) coordinates of the focus \((1, 0)\) and equation of the directrix \(x = -1\)
(c) coordinates of the focus \((-4, 0)\) and equation of the directrix \(x = 4\)
(d) coordinates of the focus \((3, 0)\) and coordinates of the vertex \((0, 0)\)
(e) coordinates of the vertex \((0, 0)\) equation of the axis \(y = 0\) and focal length 9
(f) coordinates of the vertex \((0, 0)\), equation of the axis \(y = 0\) and focal length 2
(g) coordinates of the vertex \((0, 0)\) and equation of the axis \(y = 0\) and passing through the point \((3, 6)\)
(h) coordinates of the vertex \((0, 0)\) and equation of the axis \(y = 0\) and passing through the point \((2, 1)\).

6. Find the coordinates of the focus, the equation of the directrix and the focal length of the parabola
(a) \( y^2 = 8x \)
(b) \( y^2 = 4x \)
(c) \( y^2 = -12x \)
(d) \( y^2 = 6x \)
(e) \( y^2 = -5x \)
(f) \( 3y^2 = x \)

7. Find the equation of the focal chord that cuts the curve \(y^2 = 16x\) at \((4, 8)\).

8. Find the length of the latus rectum of the parabola \(y^2 = 12x\). What are the coordinates of its endpoints?

9. The line with equation \(x - 3y - 27 = 0\) meets the parabola \(y^2 = 4x\) at two points. Find their coordinates.

10. Let \(R\left(\frac{1}{5}, -2\right)\) be a point on the parabola \(y^2 = 20x\).
(a) Find the equation of the focal chord passing through \(R\).
(b) Find the coordinates of the point \(Q\) where this chord cuts the directrix.
(c) Find the area of \(\Delta OFQ\) where \(O\) is the origin and \(F\) is the focus.
(d) Find the perpendicular distance from the chord to the point \(P(-1, -7)\).
(e) Hence find the area of \(\Delta PQR\).
Application

A parabolic satellite dish receives its signals through the focus. If the dish has height 12 m and a span of 20 m, find where the focus should be placed, to the nearest mm.

**SOLUTION**

The parabola is of the form \( x^2 = 4ay \) and passes through (10, 12) and (−10, 12)
Substituting (10, 12) gives

\[
10^2 = 4a (12) \\
100 = 48a \\
2.083 = a
\]

So the focus should be placed 2.083 m from the vertex.

Here is a summary of the 4 different types of parabola with the vertex at the origin.

1. \( x^2 = 4ay \)
2. \( x^2 = -4ay \)

![Diagram showing the parabola with vertex at (0, -a) and directrix \( y = a \).]

3. \( y^2 = 4ax \)

![Diagram showing the parabola with vertex at (a, 0) and directrix \( x = -a \).]

4. \( y^2 = -4ax \)

![Diagram showing the parabola with vertex at (-a, 0) and directrix \( x = a \).]

**General Parabola**

When the parabola does not have its vertex at the origin, there is a more general formula.

Since we use \( a \) to mean the focal length, we cannot use \((a, b)\) as the vertex. We use \((h, k)\) instead.
PARABOLA \( (x - h)^2 = 4a(y - k) \)

The concave upwards parabola with vertex \((h, k)\) and focal length \(a\) has equation \((x - h)^2 = 4a(y - k)\)

**Proof**

Find the equation of the parabola with vertex \((h, k)\) and focal length \(a\).

Counting up \(a\) units from vertex \(V\) gives the focus \(F = (h, k + a)\).

Counting down \(a\) units from \(V\) gives the point on the directrix \(D = (h, k - a)\).

So the equation of the directrix is given by \(y = k - a\).

We find the equation of the locus of \(P(x, y)\) that is equidistant from point \(F(h, k + a)\) and line \(y = k - a\).

\(B\) has coordinates \((x, k - a)\).

We want \(PF = PB\)

i.e. \( PF^2 = PB^2 \)

\((x - h)^2 + [y - (k + a)]^2 = (x - x)^2 + [y - (k + a)]^2 \)

\((x - h)^2 + (y - k - a)^2 = (y - k + a)^2 \)

\((x - h)^2 = (y - k + a)^2 - (y - k - a)^2 \)

\[= [(y - k + a) + (y - k - a)] \times [(y - k + a) - (y - k - a)] \]

\[= (2y - 2k)(2a) \]

\[= 4ay - 4ak \]

\[= 4a(y - k) \]
The parabola \((x - h)^2 = 4a(y - k)\) has
- **axis** parallel to the \(y\)-axis
- **vertex** at \((h, k)\)
- **focus** at \((h, k + a)\)
- **directrix** with equation \(y = k - a\)

**EXAMPLES**

1. Find the equation of the parabola with focus \((2, 3)\) and directrix with equation \(y = -7\).

   **Solution**

   Coordinates of \(B\) are \((2, -7)\).
   The vertex is the midpoint of \((2, 3)\) and \((2, -7)\).
   \[\therefore \text{vertex} = (2, -2)\]
   
   Focal length is the distance from the focus to the vertex.
   \[\therefore \quad a = 5\]
   
   From the diagram the parabola is concave upwards.
   The equation is in the form
   \[(x - h)^2 = 4a(y - k)\]
   i.e.
   \[\begin{align*}
   (x - 2)^2 &= 4(5)[y - (-2)] \\
   &= 20(y + 2) \\
   x^2 - 4x + 4 &= 20y + 40 \\
   x^2 - 4x - 20y - 36 &= 0
   \end{align*}\]

2. Find the coordinates of the vertex and the focus, and the equation of the directrix, of the parabola with equation \(x^2 + 6x - 12y - 3 = 0\).
Solution

Complete the square on $x$.

\[ x^2 + 6x - 12y - 3 = 0 \]

\[ x^2 + 6x = 12y + 3 \]

\[ x^2 + 6x + 9 = 12y + 3 + 9 \]

\[ (x + 3)^2 = 12y + 12 \]

\[ = 12(y + 1) \]

So the parabola has equation \((x + 3)^2 = 12(y + 1)\).

Its vertex has coordinates \((-3, -1)\).

\[ 4a = 12 \]

\[ \therefore \ a = 3 \]

The parabola is concave upwards as it is in the form \((x - h)^2 = 4a(y - k)\).

Count up 3 units to the focus.

\[ \therefore \ \text{focus} = (-3, 2) \]

Count down 3 units to the directrix.

\[ \therefore \ \text{directrix has equation} \ y = -4. \]

PARABOLA \((x - h)^2 = -4a(y - k)\)

The concave downwards parabola with vertex \((h, k)\) and focal length \(a\) has equation \((x - h)^2 = -4a(y - k)\).

Proof

Find the equation of the concave downwards parabola with vertex \((h, k)\) and focal length \(a\).
Counting down \(a\) units from the vertex \(V\) gives the focus \(F = (h, k - a)\). Counting up \(a\) units from the vertex \(V\) gives the point on the directrix \(D = (h, k + a)\).

So the equation of the directrix is given by \(y = k + a\).

We find the equation of the locus of \(P(x, y)\) that is equidistant from point \(F(h, k - a)\) and line \(y = k + a\).

\[
B\text{ has coordinates } (x, k + a).
\]

We want \(PF = PB\)

\[
PF^2 = PB^2
\]

\[
(x - h)^2 + [y - (k - a)]^2 = (x - h)^2 + [y - (k + a)]^2
\]

\[
(x - h)^2 + (y - k + a)^2 = (y - k - a)^2
\]

\[
(x - h)^2 = (y - k - a)^2 - (y - k + a)^2
\]

\[
= [(y - k - a) + (y - k + a)] [(y - k - a) - (y - k + a)]
\]

(difference of two squares)

\[
= (2y - 2k)(-2a)
\]

\[
= -4ay + 4ak
\]

\[
= -4a(y - k)
\]
The parabola \((x - h)^2 = -4a(y - k)\) has
- axis parallel to the \(y\)-axis
- vertex at \((h, k)\)
- focus at \((h, k - a)\)
- directrix with equation \(y = k + a\)

**EXAMPLES**

1. Find the equation of the parabola with focus \((-2, 1)\) and directrix \(y = 3\).

**Solution**

Coordinates of \(B\) are \((-2, 3)\).
The vertex is the midpoint of \((-2, 1)\) and \((-2, 3)\).
\(\therefore\) vertex = \((-2, 2)\)
Focal length \(a = 1\).
From the diagram the curve is concave downwards.
The equation is in the form
\((x - h)^2 = -4a(y - k)\)
i.e.  
\[ \begin{align*}
(x - (-2))^2 &= -4(1)(y - 2) \\
(x + 2)^2 &= -4(y - 2) \\
x^2 + 4x + 4 &= -4y + 8 \\
x^2 + 4x + 4y - 4 &= 0.
\end{align*} \]

2. Find the coordinates of the vertex and focus, and the equation of the directrix of the parabola \(x^2 - 8x + 8y - 16 = 0\).
Solution

Complete the square on $x$.

\[ x^2 - 8x + 8y - 16 = 0 \]

\[ x^2 - 8x = -8y + 16 \]

\[ x^2 - 8x + 16 = -8y + 16 + 16 \]

\[ (x - 4)^2 = -8y + 32 \]

\[ = -8(y - 4) \]

So the parabola has equation $(x - 4)^2 = -8(y - 4)$.

Its vertex has coordinates $(4, 4)$.

\[ 4a = 8 \]

\[ \therefore a = 2 \]

The parabola is concave downwards as it is in the form $(x - h)^2 = -4a(y - k)$.

Count down 2 units to the focus

\[ \therefore \text{focus} = (4, 2) \]

Count up 2 units to the directrix

\[ \therefore \text{directrix has equation } y = 6. \]

PARABOLA \((y - k)^2 = 4a(x - h)\)

The parabola with vertex \((h, k)\) and focal length \(a\) that turns to the right has equation \((y - k)^2 = 4a(x - h)\)

Proof

Find the equation of the parabola that turns to the right with vertex \((h, k)\) and focal length \(a\).
Counting \(a\) units to the right from the vertex \(V\) gives the focus \(F = (h + a, k)\).
Counting \(a\) units to the left from the vertex \(V\) gives the point on the directrix \(D = (h - a, k)\).
So the equation of the directrix is given by \(x = h - a\).
We find the equation of the locus of \(P(x, y)\) that is equidistant from point \(F(h + a, k)\) and line \(x = h - a\).

\[ B \text{ has coordinates } (h - a, y). \]

We want \(PF = PB\)

\[ PF^2 = PB^2 \]

\[ (x - (h + a))^2 + (y - k)^2 = (x - (h - a))^2 + (y - y)^2 \]

\[ (x - h - a)^2 + (y - k)^2 = (x - h - a)^2 \]

\[ (y - k)^2 = (x - h + a)^2 - (x - h - a)^2 \]

\[ = [(x - h + a) + (x - h - a)][(x - h + a) - (x - h - a)] \]

\[ = (2x - 2h)(2a) \]

\[ = 4ax - 4ah \]

\[ = 4a(x - h) \]
The parabola \((y - k)^2 = 4a(x - h)\) has
- **axis** parallel to the \(x\)-axis
- **vertex** at \((h, k)\)
- **focus** at \((h + a, k)\)
- **directrix** with equation \(x = h - a\)

### EXAMPLES

1. Find the equation of the parabola with focus \((1, -1)\) and directrix \(x = -5\).

**Solution**

Coordinates of \(B\) are \((-5, -1)\).
The vertex is the midpoint of \((-5, -1)\) and \((1, -1)\).
\[ \therefore \text{vertex} = (-2, -1) \]
Focal length \(a = 3\)
From the diagram the parabola curves to the right.
The equation is in the form
\[(y - k)^2 = 4a(x - h)\]
i.e. \[\left[ y - (-1) \right]^2 = 4(3)\left[ x - (-2) \right]\]
\[ (y + 1)^2 = 12(x + 2) \]
\[ y^2 + 2y + 1 = 12x + 24 \]
\[ y^2 + 2y - 12x - 23 = 0 \]
2. Find the coordinates of the vertex and focus, and the equation of the directrix of the parabola \( y^2 + 12y - 4x - 8 = 0 \).

**Solution**

Complete the square on \( y \).
\[
\begin{align*}
y^2 + 12y - 4x - 8 &= 0 \\
y^2 + 12y &= 4x + 8 \\
y^2 + 12y + 36 &= 4x + 8 + 36 \\
(y + 6)^2 &= 4x + 44 \\
&= 4(x + 11)
\end{align*}
\]
So the parabola has equation \((y + 6)^2 = 4(x + 11)\)
or \([y - (-6)]^2 = 4[x - (-11)]\).

Its vertex has coordinates \((-11, -6)\).
\[
4a = 4 \quad \therefore \quad a = 1
\]
The parabola turns to the right as it is in the form \((y - k)^2 = 4a(x - h)\).

\[
\begin{align*}
x &= -12 \\
(-11, -6) \\
(-10, -6)
\end{align*}
\]

Count 1 unit to the right for the focus
\(\therefore\) focus = \((-10, -6)\).
Count 1 unit to the left for the directrix
\(\therefore\) directrix has equation \(x = -12\).

**PARABOLA** \((y - k)^2 = -4a(x - h)\)

The parabola with vertex \((h, k)\) and focal length \(a\) that turns to the left has equation \((y - k)^2 = -4a(x - h)\)
**Proof**

Find the equation of the parabola that turns to the left with vertex \((h, k)\) and focal length \(a\).

Counting \(a\) units to the left from the vertex \(V\) gives the focus \(F = (h - a, k)\).

Counting \(a\) units to the right from the vertex \(V\) gives the point on the directrix \(D = (h + a, k)\).

So the equation of the directrix is given by \(x = h + a\).

We find the equation of the locus of \(P(x, y)\) that is equidistant from point \(F(h - a, k)\) and line \(x = h + a\).

\[P\text{F} = PB\]

\[PF^2 = PB^2\]

\[\begin{align*}
[x - (h - a)]^2 + (y - k)^2 &= [x - (h + a)]^2 + (y - y)^2 \\
(x - h + a)^2 + (y - k)^2 &= (x - h - a)^2 \\
(y - k)^2 &= (x - h - a)^2 - (x - h + a)^2 \\
&= [(x - h - a) + (x - h + a)][(x - h - a) - (x - h + a)] \\
&= (2x - 2h)(-2a) \\
&= -4ax + 4ah \\
&= -4a(x - h) \\
&= -4a(y - k)
\end{align*}\]
The parabola \((y - k)^2 = -4a(x - h)\) has
- axis parallel to the x-axis
- vertex at \((h, k)\)
- focus at \((h - a, k)\)
- directrix with equation \(x = h + a\)

EXAMPLES

1. Find the equation of the parabola with focus (2, 1) and directrix \(x = 3\).

Solution

Coordinates of \(B\) are (3, 1).
The vertex is the midpoint of (3, 1) and (2, 1).
\[
\therefore \text{vertex} = \left(2\frac{1}{2}, 1\right)
\]
Focal length \(a = \frac{1}{2}\)

From the diagram the parabola curves to the left.
The equation is in the form
\[(y - k)^2 = -4a(x - h)\]
i.e.
\[
(y - 1)^2 = -4\left(\frac{1}{2}\right)\left(x - 2\frac{1}{2}\right)
\]
\[
(y - 1)^2 = -2\left(x - 2\frac{1}{2}\right)
\]
\[
y^2 - 2y + 1 = -2x + 5
\]
\[
y^2 - 2y + 2x - 4 = 0
\]

2. Find the coordinates of the vertex and focus, and the equation of the directrix of the parabola \(y^2 + 4y + 8x - 4 = 0\).
Solution

Complete the square on $y$.

\[
y^2 + 4y + 8x - 4 = 0
\]

\[
y^2 + 4y = -8x + 4
\]

\[
y^2 + 4y + 4 = -8x + 4 + 4
\]

\[
(y + 2)^2 = -8x + 8
\]

\[
= -8(x - 1)
\]

So the parabola has equation \((y + 2)^2 = -8(x - 1)\)

or \([y - (-2)]^2 = -8(x - 1)\).

Its vertex has coordinates \((1, -2)\).

\[4a = 8\]

\[
\therefore \ a = 2
\]

The parabola turns to the left as it is in the form \((y - k)^2 = -4a(x - h)\)

Count 2 units to the left for the focus

\[\therefore \text{ focus } = (-1, -2)\].

Count 2 units to the right for the directrix

\[\therefore \text{ directrix has equation } x = 3\].

11.6 Exercises

1. Complete the square on $x$ to write each equation in the form \((x - h)^2 = \pm 4a(y - k)\).

(a) \(x^2 - 6x - 8y - 15 = 0\)

(b) \(x^2 - 10x - 4y + 1 = 0\)

(c) \(x^2 - 2x - 4y - 11 = 0\)

(d) \(x^2 - 8x + 12y - 20 = 0\)

(e) \(x^2 - 12x - 8y - 20 = 0\)

(f) \(x^2 + 14x + 16y + 1 = 0\)

(g) \(x^2 - 4x + 4y - 16 = 0\)

(h) \(x^2 + 18x - 12y + 9 = 0\)

(i) \(x^2 + 2x - 8y - 7 = 0\)

(j) \(x^2 - 6x + 4y + 1 = 0\)
2. Complete the square on $y$ to write each equation in the form $(y-k)^2 = 4a(x-h)$

(a) $y^2 - 8y - 4x = 0$
(b) $y^2 - 2y - 8x - 15 = 0$
(c) $y^2 + 4y + 12x - 8 = 0$
(d) $y^2 - 20y + 4x - 16 = 0$
(e) $y^2 + 6y + 16x - 7 = 0$
(f) $y^2 - 12y - 8x + 4 = 0$
(g) $y^2 + 10y + 24x - 23 = 0$
(h) $y^2 + 24y - 4x = 0$
(i) $y^2 - 4y + 20x - 16 = 0$
(j) $y^2 + 8y + 8x = 0$

3. Find the equation of each parabola

(a) focus $(-1, 3)$, directrix $y = -1$
(b) focus $(-4, 1)$, directrix $y = -1$
(c) focus $(2, 0)$, directrix $y = -4$
(d) focus $(3, 6)$, directrix $y = 2$
(e) focus $(-2, 5)$, directrix $y = -3$
(f) focus $(-1, -4)$, directrix $y = 4$
(g) focus $(4, -3)$, directrix $y = 7$
(h) focus $(-5, 1)$, directrix $y = 5$
(i) focus $(-3, -6)$, directrix $y = 0$
(j) focus $(0, -7)$, directrix $y = -5$
(k) focus $(2, 3)$, directrix $x = -4$
(l) focus $(-1, 4)$, directrix $x = -3$
(m) focus $(6, 0)$, directrix $x = 2$
(n) focus $(3, -2)$, directrix $x = -5$
(o) focus $(1, -1)$, directrix $x = -3$
(p) focus $(-2, -4)$, directrix $x = 4$
(q) focus $(2, 1)$, directrix $x = 4$
(r) focus $(-5, 3)$, directrix $x = 3$
(s) focus $(-1, 2)$, directrix $x = 0$
(t) focus $(3, 1)$, directrix $x = 4$

4. Find

(i) the coordinates of the focus and
(ii) the equation of the directrix of each parabola

(a) $x^2 - 6x - 4y - 3 = 0$
(b) $x^2 - 2x - 8y - 7 = 0$
(c) $x^2 + 4x - 4y = 0$
(d) $x^2 - 8x - 12y + 4 = 0$
(e) $x^2 + 10x - 8y + 1 = 0$
(f) $x^2 - 6x + 4y + 1 = 0$
(g) $x^2 + 2x + 8y - 15 = 0$
(h) $x^2 - 4x + 4y = 0$
(i) $x^2 - 8x + 12y + 4 = 0$
(j) $x^2 + 4x + 16y - 12 = 0$

5. Find

(i) the coordinates of the focus and
(ii) the equation of the directrix of each parabola

(a) $y^2 + 2y - 4x - 3 = 0$
(b) $y^2 - 8y - 12x + 4 = 0$
(c) $y^2 - 6y - 8x - 7 = 0$
(d) $y^2 + 4y - 16x - 12 = 0$
(e) $y^2 - 2y - 24x + 25 = 0$
(f) $y^2 + 10y + 8x + 1 = 0$
(g) $y^2 + 14y + 4x + 1 = 0$
(h) $y^2 - 12y + 20x - 4 = 0$
(i) $y^2 - 4y + 32x - 28 = 0$
(j) $y^2 + 6y + 40x + 29 = 0$

6. Find the equation of the parabola with vertex $(0, 3)$ if it is concave upwards and $a = 3$.

7. Find the equation of the parabola with vertex $(-2, -1)$, focal length 2, and axis parallel to the $y$-axis.

8. A parabola has its vertex at $(1, -5)$ and its focal length as 1. If the parabola is concave upwards, find its equation.

9. A parabola has its axis parallel to the $x$-axis. If its vertex has coordinates $(2, 6)$ and $a = 3$, find its equation if it turns to the left.

10. Find the equation of the parabola with vertex at $(1, 0)$ and focus at $(-1, 4)$.

11. Find the equation of the parabola that has vertex $(1, 1)$ and focus $(1, 8)$.

12. A parabola has its vertex at $(2, -2)$ and focus at $(-4, -2)$. Find its equation.
13. Find the equation of the parabola with vertex \((0, 3)\) and focus \((8, 3)\).

14. Find the equation of the parabola with vertex \((3, 3)\) and equation of directrix \(y = 5\).

15. Find the equation of the parabola with vertex \((3, -1)\) and directrix \(x = -1\).

16. A parabola has directrix \(y = 5\) and focus \((-3, 3)\). Find its equation.

17. Find the equation of the locus of a point moving so that it is equidistant from the point \((2, 2)\) and the line \(y = -4\).

18. Find the equation of the parabola with focus \((2, -1)\) and directrix \(x = 10\).

19. Find the coordinates of the vertex and focus and the equation of the directrix for the parabola
(a) \(x^2 + 4x - 8y + 12 = 0\)
(b) \(x^2 - 6x - 12y + 33 = 0\)
(c) \(x^2 - 2x + 4y + 5 = 0\)
(d) \(y^2 - 8y - 16x + 64 = 0\)
(e) \(y^2 + 4y - 24x + 4 = 0\)
(f) \(y^2 + 8x + 40 = 0\).

20. For the parabola \(x^2 + 2x + 28y - 111 = 0\), find the coordinates of its vertex and focus, and the equations of its directrix and axis. What is its maximum value?

21. The latus rectum of a parabola has endpoints \((-2, 3)\) and \((6, 3)\). Find two possible equations for the parabola.

22. (a) Find the equation of the arch above.
(b) Find the coordinates of its focus and the equation of its directrix.

23. (a) Sketch \(y = x^2 + 2x - 8\), showing intercepts and the minimum point.
(b) Find the coordinates of the focus and the equation of the directrix of the parabola.

24. Find the equation of the parabola with vertex \((-2, 3)\) that also passes through \((2, 1)\) and is concave downwards.

25. A parabolic satellite dish has a diameter of 4 m at a depth of 0.4 m. Find the depth at which its diameter is 3.5 m, correct to 1 decimal place.

DID YOU KNOW?

The word ‘directrix’ is due to the Dutch mathematician Jan De Witt (1629–72). He published a work called *Elementa curvarum*, in which he defined the properties of the parabola, ellipse, circle and hyperbola. These curves are all called conic sections.
De Witt was well known as the ‘Grand Pensionary of Holland’. He took part in the politics and wars of his time, opposing Louis XIV. When the French invaded Holland in 1672, De Witt was seized and killed.

**Tangents and Normals**

Remember that the gradient of the tangent to a curve is given by the derivative.

The **normal** to the curve is **perpendicular to its tangent** at that point. That is, \( m_1 m_2 = -1 \) for perpendicular lines.

**EXAMPLES**

1. Find the gradient of the tangent to the parabola \( x^2 = 8y \) at the point \((4, 2)\).

**Solution**

\[
\begin{align*}
  x^2 &= 8y \\
  \therefore y &= \frac{x^2}{8} \\
  \frac{dy}{dx} &= \frac{2x}{8} \\
  &= \frac{x}{4}
\end{align*}
\]
1. Find the gradient of the tangent to the parabola \( x^2 = 12y \) at the point where \( x = 2 \).

2. Find the gradient of the tangent to the parabola \( x^2 = -3y \) at the point \((6, -12)\).

3. Find the gradient of the normal to the parabola \( x^2 = 4y \) at the point where \( x = 2 \).

4. Find the gradient of the tangent to the parabola \( x^2 = 16y \) at the point \((4, 1)\).
5. Show that the gradient of the tangent to the curve \( x^2 = 2y \) at any point is its \( x \)-coordinate.

6. Find the equation of the tangent to the curve \( x^2 = 8y \) at the point \((4, 2)\).

7. Find the equation of the normal to the curve \( x^2 = 4y \) at the point where \( x = -4 \).

8. Find the equations of the tangent and normal to the parabola \( x^2 = -24y \) at the point \((12, -6)\).

9. Find the equations of the tangent and normal to the parabola \( x^2 = 16y \) at the point where \( x = 4 \).

10. Find the equation of the tangent to the curve \( x^2 = -2y \) at the point \((4, -8)\). This tangent meets the directrix at point \( M \). Find the coordinates of \( M \).

11. Find the equation of the normal to the curve \( x^2 = 12y \) at the point \((6, 3)\). This normal meets the parabola again at point \( P \). Find the coordinates of \( P \).

12. The normal of the parabola \( x^2 = 18y \) at \((-6, 2)\) cuts the parabola again at \( Q \). Find the coordinates of \( Q \).

13. Find the equations of the normals to the curve \( x^2 = -8y \) at the points \((-16, -32)\) and \((-2, -\frac{1}{2})\). Find their point of intersection and show that this point lies on the parabola.

14. Find the equation of the tangent at \((8, 4)\) on the parabola \( x^2 = 16y \). This tangent meets the tangent at the vertex of the parabola at point \( R \). Find the coordinates of \( R \).

15. (a) Show that the point \( P(2p, p^2) \) lies on the parabola \( x^2 = 4y \). (b) Find the equation of the normal to the parabola at \( P \). (c) Show that \( p^2 + 1 = 0 \) if the normal passes through the focus of the parabola \( (p \neq 0) \).

### Parametric Equations of the Parabola

An equation involving \( x \) and \( y \), for example \( x^2 = 4ay \), is called a **Cartesian equation**.

Equations can also be written in **parametric form**. In this form, \( x \) and \( y \) are both written in terms of a **third variable** called a **parameter**.

An example of a Cartesian equation is \( y = x^2 - 1 \).

An example of parametric equations is \( x = 2t + 3, \ y = t - 2 \).

Any Cartesian equation can be written in parametric form.
EXAMPLE

Write \( y = 3x + 1 \) in parametric form.

Solution

There are many different ways this can be done.
For example: Given parameter \( p \)
(a) \( \text{Let } x = p \)
Then \( y = 3x + 1 \)
\( = 3p + 1 \)
So parametric equations are \( x = p, y = 3p + 1 \).
(b) \( \text{Let } x = p - 5 \)
Then \( y = 3x + 1 \)
\( = 3(p - 5) + 1 \)
\( = 3p - 15 + 1 \)
\( = 3p - 14 \)
So parametric equations are \( x = p - 5, y = 3p - 14 \).

There are many different ways to write parametric equations. Can you find some more for the example above?
We can also change parametric equations back into Cartesian form.

EXAMPLES

1. Find the Cartesian equation of \( x = 3t + 1, y = 2t - 3 \).

Solution

We use the process for solving simultaneous equations to eliminate the parameter.
\[
\begin{align*}
x &= 3t + 1 \\
y &= 2t - 3
\end{align*}
\]
\[
\begin{align*}
\text{From (1)} \\
x - 1 &= 3t \\
\frac{x - 1}{3} &= t
\end{align*}
\]
Substitute in (2)
\[
\begin{align*}
y &= 2t - 3 \\
&= 2 \left( \frac{x - 1}{3} \right) - 3 \\
3y &= 2(x - 1) - 9 \\
&= 2x - 2 - 9 \\
&= 2x - 11 \\
0 &= 2x - 3y - 11
\end{align*}
\]
2. Find the Cartesian equation of \( x = 2q, y = q^2 - 3 \).

**Solution**

\[
\begin{align*}
x &= 2q \quad (1) \\
y &= q^2 - 3 \quad (2)
\end{align*}
\]

From (1)
\[
x = 2q
\]
\[
\frac{x}{2} = q
\]

Substitute in (2)
\[
y = q^2 - 3
\]
\[
= \left( \frac{x}{2} \right)^2 - 3
\]
\[
= \frac{x^2}{4} - 3
\]
\[
4y = x^2 - 12
\]
\[
0 = x^2 - 4y - 12
\]

The equation of a parabola can be written as a set of parametric equations.

The parabola \( x^2 = 4ay \) can be written as
\[
\begin{align*}
x &= 2at \\
y &= at^2
\end{align*}
\]
where \( t \) is a parameter.

**Proof**

Substitute \( x = 2at \) into \( x^2 = 4ay \)
\[
(2at)^2 = 4ay
\]
\[
4a^2t^2 = 4ay
\]
\[
at^2 = y
\]
\[
\therefore \ x = 2at \text{ and } y = at^2 \text{ satisfy the equation } x^2 = 4ay
\]

**Class Investigation**

1. How would you write \( x^2 = -4ay \) in parametric form?
2. How would you write \( y^2 = 4ax \) in parametric form?
3. How would you write \( y^2 = -4ax \) in parametric form?
The parabola $x^2 = -4ay$ can be written as
\[
\begin{align*}
  x &= 2at \\
  y &= -at^2 
\end{align*}
\]

**Proof**

Substitute $x = 2at$ into $x^2 = -4ay$

\[
(2at)^2 = -4ay \\
4a^2t^2 = -4ay \\
at^2 = -y \\
-at^2 = y
\]

\[
\therefore \ x = 2at \text{ and } y = -at^2 \text{ satisfy the equation } x^2 = -4ay.
\]

The parabola $y^2 = 4ax$ can be written as
\[
\begin{align*}
  x &= at^2 \\
  y &= 2at 
\end{align*}
\]

**Proof**

Substitute $y = 2at$ into $y^2 = 4ax$

\[
(2at)^2 = 4ax \\
4a^2t^2 = 4ax \\
at^2 = x
\]

\[
\therefore \ x = at^2 \text{ and } y = 2at \text{ satisfy the equation } y^2 = 4ax.
\]

The parabola $y^2 = -4ax$ can be written as
\[
\begin{align*}
  x &= -at^2 \\
  y &= 2at 
\end{align*}
\]

**Proof**

Substitute $y = 2at$ into $y^2 = -4ax$

\[
(2at)^2 = -4ax \\
4a^2t^2 = -4ax \\
at^2 = -x \\
-at^2 = x
\]

\[
\therefore \ x = -at^2 \text{ and } y = 2at \text{ satisfy the equation } y^2 = -4ax.
\]
EXAMPLES

1. Given the parabola \( x = 4t \) and \( y = 2t^2 \), find
   (a) its Cartesian equation
   (b) the points on the parabola when \( t = \pm 2 \).

Solution

(a) \( x = 4t \)

\[ \therefore \frac{x}{4} = t \]

Substitute into \( y = 2t^2 \):

\[ y = 2\left(\frac{x}{4}\right)^2 \]

\[ = \frac{2x^2}{16} \]

\[ = \frac{x^2}{8} \]

\[ 8y = x^2 \]

(b) When \( t = 2 \)

\[ x = 4(2) \]

\[ = 8 \]

\[ y = 2(2)^2 \]

\[ = 8 \]

When \( t = -2 \)

\[ x = 4(-2) \]

\[ = -8 \]

\[ y = 2(-2)^2 \]

\[ = 8 \]

\[ \therefore \text{points are } (8, 8) \text{ and } (-8, 8). \]

2. Find the coordinates of the focus and the equation of the directrix of the parabola \( x = -12t \), \( y = -6t^2 \).

Solution

Method 1:

We can find the Cartesian equation.

\[ x = -12t \] (1)

\[ y = -6t^2 \] (2)

From (1)

\[ x = -12t \]

\[ \frac{x}{-12} = t \]
Substitute in (2)
\[
y = -6t^2
= -6\left(-\frac{x}{12}\right)^2
= -6\left(-\frac{x^2}{144}\right)
= -\frac{x^2}{24}
\]
\[
-24y = x^2
\]
This is in the form \( x^2 = -4ay \) (concave downwards parabola with vertex at the origin).
\[
4a = 24
a = 6
\]
So focal length is 6 units.

Method 2:
The equations \( x = -12t, y = -6t^2 \) are in the form \( x = -2at, y = -at^2 \).
\[
\therefore a = 6
\]
The equations satisfy \( x^2 = -4ay \)
\[
x^2 = -4(6)y
= -24y
\]
This is a concave downward parabola with focus \((0, -a)\) and directrix \( y = a \).
So focus = \((0, -6)\) and directrix has equation \( y = 6 \).

3. Write \( x^2 = 32y \) as a set of parametric equations.

Solution

\[
4a = 32
\]
So \( a = 8 \)

Equations are in the form \( x = 2at, y = at^2 \).
\[
\text{So } x = 2(8)t, y = 8t^2
\]
\[
x = 16t, \quad y = 8t^2
\]
4. Write \( y^2 = 12x \) in parametric form.

**Solution**

\[ 4a = 12 \]
\[ a = 3 \]

Equations are in the form \( x = at^2, y = 2at \)

So \( x = 3t^2, y = 2(3)t \)
\[ x = 3t^2, y = 6t \]

### 11.8 Exercises

1. Sketch the graph of
   (a) \( x = t - 2, y = t^2 \)
   (b) \( x = t - 2, y = 3t - 1 \)
   (c) \( x = 2t, y = 4t - 3 \)
   (d) \( x = t + 1, y = t^2 \)
   (e) \( x = 2t, y = 2t^2 - 3 \)
   (f) \( x = 6t, y = 3t^2 \)

2. Find the Cartesian equation of
   (a) \( x = 4t, y = 2t - 1 \)
   (b) \( x = t + 3, y = 2t - 5 \)
   (c) \( x = t - 1, y = t^2 + t \)
   (d) \( x = \frac{t}{2}, y = 4t^2 - 1 \)
   (e) \( x = \frac{1}{t}, y = 2t \)

3. Write as a set of parametric equations
   (a) \( x^2 = 4y \)
   (b) \( x^2 = 12y \)
   (c) \( x^2 = -8y \)
   (d) \( x^2 = 16y \)
   (e) \( x^2 = -36y \)
   (f) \( x^2 = 20y \)
   (g) \( x^2 = -6y \)
   (h) \( x^2 = y \)
   (i) \( 2x^2 = y \)
   (j) \( x^2 = -10y \)

4. Find the Cartesian equation for each parabola
   (a) \( x = 8t, y = 4t^2 \)
   (b) \( x = 10t, y = 5t^2 \)
   (c) \( x = 2t, y = t^2 \)
   (d) \( x = -14t, y = -7t^2 \)
   (e) \( x = 4t, y = -2t^2 \)
   (f) \( x = 2at, y = at^2 \)
   (g) \( x = 2m, y = -m^2 \)
   (h) \( x = 12p, y = 6p^2 \)
   (i) \( x = -t, y = -\frac{1}{2}t^2 \)
   (j) \( x = 2aq, y = aq^2 \)

5. (a) Show that \( (6t, -3t^2) \) lies on the parabola \( x^2 = -12y \) for all values of \( t \).
   (b) Find the coordinates of point \( P \) where \( t = -2 \).
   (c) Find the equation of the tangent to the parabola at \( P \).

6. (a) Find the coordinates of \( Q \) on the parabola \( x = 8t, y = 4t^2 \) at the point where \( t = -1 \).
   (b) Find the equation of the normal to the parabola at \( Q \).
7. A parabola has equations
   \[ x = 4t^2, \ y = 8t. \] Find the coordinates of its focus and the
equation of its directrix.

8. Find the coordinates of point \( P \) on the parabola \( x = t^2, \ y = -2t \)
   where \( t = 2 \). Find the equation of line \( PS \) where \( S \) is the focus of the
parabola.

9. (a) Find the Cartesian equation
    of the parabola \( x = 12t, \ y = 6t^2. \)
    (b) The point \( \left( 3, \frac{3}{8} \right) \) lies on the
parabola. What is the value of \( t \) at this point?

10. Find the equation of the tangent
to the parabola \( x = 4t, \ y = 2t^2 \) at
the point where \( t = 3. \)

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**Chords, Tangents and Normals**

If \( P(2ap, ap^2) \) and \( Q(2aq, aq^2) \) are any two points on the parabola \( x^2 = 4ay, \)
then the chord \( PQ \) has gradient \( \frac{p + q}{2} \)
and equation \( y - \frac{1}{2}(p + q)x + apq = 0 \)

**Proof**

\[
m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{ap^2 - aq^2}{2ap - 2aq} = \frac{a(p^2 - q^2)}{2a(p - q)} = \frac{a(p + q)(p - q)}{2a(p - q)} = \frac{p + q}{2}
\]

The equation formula is
\[ y - y_1 = m(x - x_1) \]
\[ \therefore \ y - aq^2 = \frac{p + q}{2}(x - 2aq) \]
\[ = \frac{1}{2}(p + q)x - aq(p + q) \]
\[ = \frac{1}{2}(p + q)x - apq - aq^2 \]
\[ \therefore \ y - \frac{1}{2}(p + q)x + apq = 0 \]

Learn to derive these equations rather than memorise them.
If PQ is a focal chord, then $pq = -1$

**Proof**

$x^2 = 4ay$ has focus $(0, a)$.

PQ has equation $y - \frac{1}{2}(p + q)x + apq = 0$.

For PQ to be a focal chord, it passes through $(0, a)$.

i.e. $a - \frac{1}{2}(p + q) \cdot 0 + apq = 0$

$a + apq = 0$

$apq = -a$

$pq = -1$

The tangent to the parabola $x^2 = 4ay$ at the point $P(2ap, ap^2)$ has gradient $p$ and equation given by $y - px + ap^2 = 0$.

**Proof**

$x^2 = 4ay$

$\therefore y = \frac{x^2}{4a}$

\[
\frac{dy}{dx} = \frac{2x}{4a} = \frac{x}{2a}
\]

At $P(2ap, ap^2)$

\[
\frac{dy}{dx} = \frac{2ap}{2a} = p
\]

The equation formula is

$y - y_1 = m(x - x_1)$

$\therefore y - ap^2 = p(x - 2ap)$

$= px - 2ap^2$

$\therefore y - px + ap^2 = 0$

The tangents to the parabola $x^2 = 4ay$ at points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ intersect at the point $[a(p + q), apq]$.

**Proof**

Equation of tangent at $P$ is $y - px + ap^2 = 0$ \hspace{1cm} (1)

Equation of tangent at $Q$ is $y - qx + aq^2 = 0$ \hspace{1cm} (2)
(1) – (2).
\[-px + qx + ap^2 - aq^2 = 0\]
\[x(q - p) = a(q^2 - p^2)\]
\[= a(q + p)(q - p)\]
\[x = a(q + p)\]

Substitute in (1):
\[y - p \cdot a(q + p) + ap^2 = 0\]
\[y - apq - ap^2 + ap^2 = 0\]
\[y = apq\]
\[\therefore\text{ point of intersection is }[a(p + q), apq]\]

The normal to the curve \(x^2 = 4ay\) at point \(P(2ap, ap^2)\) has gradient \(-\frac{1}{p}\)
and equation given by \(x + py = ap^3 + 2ap\)

**Proof**

Tangent at \(P\) has gradient \(p\).
For perpendicular lines, \(m_1m_2 = -1\)
\[\therefore\text{ normal has gradient }-\frac{1}{p}\text{.}\]
The equation formula is

\[y - y_1 = m(x - x_1)\]
\[\therefore\quad y - ap^2 = -\frac{1}{p}(x - 2ap)\]
\[p(y - ap^2) = -(x - 2ap)\]
\[py - ap^3 = -x + 2ap\]
\[x + py = ap^3 + 2ap\]

The normals to the parabola \(x^2 = 4ay\) at point \(P(2ap, ap^2)\) and
\(Q(2aq, aq^2)\) intersect at \([-apq(p + q), a(p^2 + pq + q^2 + 2)]\)

**Proof**

Equation of normal at \(P\) is \(x + py = ap^3 + 2ap\) \(\text{(1)}\)
Equation of normal at \(Q\) is \(x + qy = aq^3 + 2aq\) \(\text{(2)}\)

(1) – (2):
\[py - qy = ap^3 - aq^3 + 2ap - 2aq\]
\[y(p - q) = a(p^3 - q^3) + 2a(p - q)\]
\[= a(p - q)(p^2 + pq + q^2) + 2a(p - q)\]
\[y = a(p^2 + pq + q^2) + 2a\]
\[= a(p^2 + pq + q^2 + 2)\]
Substitute in (1):
\[
x + p \cdot a(p^2 + pq + q^2 + 2) = ap^3 + 2ap
\]
\[
x + ap^2 + apq + 2ap = ap^3 + 2ap
\]
\[
x = -ap^2q - apq^2
\]
\[
= -apq(p + q)
\]
\[
\therefore \text{point of intersection is } [-apq(p + q), a(p^2 + pq + q^2 + 2)]
\]

**EXAMPLES**

1. Find the equation of the chord joining points where \( t = 3 \) and \( t = -2 \) on the parabola \( x = 2at, y = at^2 \).

   **Solution**

   When \( t = 3 \)
   \[
x = 2a(3) \quad y = a(3)^2
\]
   \[
= 6a \quad = 9a
\]
   When \( t = -2 \)
   \[
x = 2a(-2) \quad y = a(-2)^2
\]
   \[
= -4a \quad = 4a
\]
   \[
\therefore \text{points are} \ (6a, 9a) \text{ and } (-4a, 4a)
\]
   Gradient \( m = \frac{y_2 - y_1}{x_2 - x_1} \)
   \[
= \frac{4a - 9a}{-4a - 6a}
\]
   \[
= -\frac{5a}{10a}
\]
   \[
= \frac{1}{2}
\]
   The equation formula is
   \[
y - y_1 = m(x - x_1)
\]
   \[
\therefore y - 4a = \frac{1}{2}(x + 4a)
\]
   \[
2y - 8a = x + 4a
\]
   \[
0 = x - 2y + 12a
\]

2. Find the equation of the tangent to the parabola \( x^2 = 8y \) at the point \( (4t, 2t^2) \).

   **Solution**

   \[
x^2 = 8y
\]
   \[
\therefore y = \frac{x^2}{8}
\]
\[
\frac{dy}{dx} = \frac{2x}{8} = \frac{x}{4}
\]
At \((4t, 2t^2)\)
\[
\frac{dy}{dx} = \frac{4t}{4} = t
\]
The equation formula is
\[
y - y_1 = m(x - x_1)
\]
\[
\therefore y - 2t^2 = t(x - 4t) = tx - 4t^2
\]
\[
\therefore 0 = tx - y - 2t^2
\]

The equations of the tangent, normal and chord can also be derived from points in Cartesian form rather than parametric form.

If point \(A(x_1, y_1)\) lies on the parabola \(x^2 = 4ay\), then the equation of the tangent at \(A\) is given by
\[
x_{1} = 2a(y + y_1)
\]

**Proof**

\[
y = \frac{x^2}{4a}
\]
\[
\frac{dy}{dx} = \frac{2x}{4a} = \frac{x}{2a}
\]
At \((x_1, y_1)\)
\[
\frac{dy}{dx} = \frac{x_1}{2a}
\]
The equation formula is
\[
y - y_1 = m(x - x_1)
\]
\[
= \frac{x_1}{2a}(x - x_1)
\]
\[
2a(y - y_1) = x_1(x - x_1)
\]
\[
2ay - 2ay_1 = xx_1 - x_1^2 = xx_1 - 4ay_1 \quad \text{(since } x_1^2 = 4ay_1)\]
\[
2ay + 2ay_1 = xx_1
\]
\[
2a(y + y_1) = xx_1
\]
If point \( A(x_1, y_1) \) lies on the parabola \( x^2 = 4ay \), then the equation of the normal at \( A \) is given by

\[
y - y_1 = -\frac{2a}{x_1}(x - x_1)
\]

**Proof**

\[
y = \frac{x^2}{4a}
\]

\[
\frac{dy}{dx} = \frac{2x}{4a}
\]

\[
= \frac{x}{2a}
\]

At \((x_1, y_1)\)

\[
\frac{dy}{dx} = \frac{x_1}{2a}
\]

For normal, \( m_1m_2 = -1 \)

\[
\therefore \quad m_2 = -\frac{2a}{x_1}
\]

The equation formula is

\[
y - y_1 = m(x - x_1)
\]

\[
\therefore \quad y - y_1 = -\frac{2a}{x_1}(x - x_1)
\]

The equation of the chord of contact \( XY \) of tangents drawn from external point \((x_1, y_1)\) to the parabola \( x^2 = 4ay \) is given by

\[
xx_1 = 2a(y + y_1)
\]

**Proof**

Let \( X = (2ap, ap^2) \) and \( Y = (2aq, aq^2) \).

Then chord \( XY \) has equation

\[
y - \frac{1}{2}(p + q)x + apq = 0.
\]
Now the tangents at \(X\) and \(Y\) intersect at \(P(x_1, y_1)\).

But the intersection of tangents is \([a(p + q), apq]\)

\[
\therefore \quad x_1 = a(p + q) \quad (1)
\]

and \(y_1 = apq\) \quad (2)

From (1):

\[
p + q = \frac{x_1}{a} \quad (3)
\]

Substituting (2) and (3) into the equation of chord \(XY\) gives

\[
y - \frac{1}{2}(x_1)\left(x + y_1\right) = 0
\]

\[
2ay - x_1x + 2ay_1 = 0
\]

\[
2a(y + y_1) - x_1x = 0
\]

\[
2a(y + y_1) = x_1x
\]

**EXAMPLE**

Tangents are drawn from the point \(\left(\frac{1}{2}, \frac{1}{2}\right)\) to the points \(P\) and \(Q\) on the parabola \(x^2 = 4y\). Find the equation of the chord of contact \(PQ\) and the coordinates of \(P\) and \(Q\).

**Solution**

\[
x^2 = 4y
\]

\[
\therefore \quad 4a = 4
\]

\[
a = 1
\]

\(PQ\) has equation \(xx_1 = 2a(y + y_1)\) where \(x_1 = \frac{1}{2}\) and \(y_1 = -\frac{1}{2}\).

\[
\frac{1}{2}x = 2\left(y - \frac{1}{2}\right)
\]

\[
= 2y - 1
\]

\[
x = 4y - 2
\]

\[
\therefore \quad x - 4y + 2 = 0
\]

is the equation of the chord of contact.

To find \(P\) and \(Q\), solve simultaneous equations.

\[
x^2 = 4y \quad (1)
\]

\[
x - 4y + 2 = 0 \quad (2)
\]

From (2):

\[
x + 2 = 4y \quad (3)
\]

Substitute into (1):

\[
x^2 = x + 2
\]

\[
x^2 - x - 2 = 0
\]

\[
(x - 2)(x + 1) = 0
\]

\[
\therefore \quad x = 2, -1
\]
Substitute $x = 2$ into (3):
\[
2 - 4y + 2 = 0 \\
4 = 4y \\
1 = y
\]
Substitute $x = -1$ into (3):
\[
-1 - 4y + 2 = 0 \\
1 = 4y \\
\frac{1}{4} = y
\]
So $P$ and $Q$ are points $(2,1)$ and $\left(-1, \frac{1}{4}\right)$.

11.9 Exercises

1. Find the
   (i) gradient and
   (ii) equation of chord $AB$ on the parabola
   (a) $x^2 = 16y$ where $A = (8t, 4t^2)$ and $B = (8n, 4n^2)$
   (b) $x^2 = 8y$ where $A = (4p, 2p^2)$ and $B = (4q, 2q^2)$
   (c) $x^2 = 12y$ where $A = (6m, 3m^2)$ and $B = (6n, 3n^2)$
   (d) $x^2 = 20y$ where $A = (10p, 5p^2)$ and $B = (10q, 5q^2)$
   (e) $x^2 = 4y$ where $A = (2a, a^2)$ and $B = (2b, b^2)$
   (f) $x^2 = -8y$ where $A = (4p, -2p^2)$ and $B = (4q, -2q^2)$
   (g) $x^2 = -24y$ where $A = (12a, -6a^2)$ and $B = (12b, -6b^2)$
   (h) $x^2 = -16y$ where $A = (-8p, -4p^2)$ and $B = (-8q, -4q^2)$
   (i) $x^2 = -4y$ where $A = (2s, -s^2)$ and $B = (2t, -t^2)$
   (j) $x^2 = -28y$ where $A = (-14p, -7p^2)$ and $B = (-14q, -7q^2)$

2. Find
   (i) the gradient of the tangent,
   (ii) the gradient of the normal,
   (iii) the equation of the tangent and
   (iv) the equation of the normal to the curve
   (a) $x^2 = 4y$ at the point $(2p, p^2)$
   (b) $x^2 = 12y$ at the point $(6q, 3q^2)$
   (c) $x^2 = 8y$ at the point $(4t, 2t^2)$
   (d) $x^2 = 20y$ at the point $(10n, 5n^2)$
   (e) $x^2 = 24y$ at the point $(12p, 6p^2)$
   (f) $x^2 = -16y$ at the point $(8k, -4k^2)$
   (g) $x^2 = -4y$ at the point $(-2q, -q^2)$
   (h) $x^2 = -8y$ at the point $(4t, -2t^2)$
   (i) $x^2 = -12y$ at the point $(-6m, -3m^2)$
   (j) $x^2 = -32y$ at the point $(16a, -8a^2)$
3. Find the point of intersection between the
   (i) tangents and
   (ii) normals to the curve
   (a) \( x^2 = 4y \) at the points \((2p, p^2)\) and \((2q, q^2)\)
   (b) \( x^2 = 16y \) at the points \((8p, 4p^2)\) and \((8q, 4q^2)\)
   (c) \( x^2 = 8y \) at the points \((4a, 2a^2)\) and \((4b, 2b^2)\)
   (d) \( x^2 = 12y \) at the points \((6s, 3s^2)\) and \((6t, 3t^2)\)
   (e) \( x^2 = 20y \) at the points \((10t, 5t^2)\) and \((10w, 5w^2)\)
   (f) \( x^2 = -24y \) at the points \((12p, -6p^2)\) and \((12q, -6q^2)\)
   (g) \( x^2 = -16y \) at the points \((8m, -4m^2)\) and \((8n, -4n^2)\)
   (h) \( x^2 = -40y \) at the points \((20p, -10p^2)\) and \((20q, -10q^2)\)
   (i) \( x^2 = -20y \) at the points \((10h, -5h^2)\) and \((10k, -5k^2)\)
   (j) \( x^2 = -12y \) at the points \((-6p, -3p^2)\) and \((-6q, -3q^2)\)

4. Find the equation of the
   (i) tangent and
   (ii) normal at the point \((x_1, y_1)\)
   to the parabola
   (a) \( x^2 = 8y \)
   (b) \( x^2 = 12y \)
   (c) \( x^2 = 16y \)
   (d) \( x^2 = 4y \)
   (e) \( x^2 = 20y \)
   (f) \( x^2 = -4y \)
   (g) \( x^2 = -8y \)
   (h) \( x^2 = -24y \)
   (i) \( x^2 = -44y \)
   (j) \( x^2 = -28y \)

5. Find the equation of the chord of contact \(AB\) of tangents drawn from an external point \((x_0, y_0)\) to the parabola
   (a) \( x^2 = 16y \)
   (b) \( x^2 = 4y \)
   (c) \( x^2 = 8y \)
   (d) \( x^2 = 12y \)
   (e) \( x^2 = 20y \)
   (f) \( x^2 = -4y \)
   (g) \( x^2 = -24y \)
   (h) \( x^2 = -8y \)
   (i) \( x^2 = -16y \)
   (j) \( x^2 = -36y \)

6. Derive the equation of the
tangent to the curve \(x^2 = 4ay\) at the point
   (a) \((2ap, ap^2)\)
   (b) \((x_0, y_0)\)

7. Find the equation of chord
   \(XY\) on the parabola \(x^2 = 8y\)
   where \(X = (4t, 2t^2)\) and \(Y = (4r, 2r^2)\).

8. Find the equation of chord \(PQ\) on the parabola \(x = 6t, y = 3t^2\), given that \(t = 2\) at \(P\) and \(t = 3\) at \(Q\).

9. Show that the equation of the
   normal to the parabola \(x^2 = -18y\)
   at the point \((-9t, -\frac{9t^2}{2})\) is given by \(2x + 2yt + 9t^3 + 18t = 0\).

10. Derive the equation of the
    normal to the parabola \(x^2 = 4ay\)
    at the point \((2at, at^2)\).

11. Find the equation of the chord of
    contact of tangents drawn from
    the external point \((3, -1)\) to the
    parabola \(x^2 = 8y\).

12. Show that \(3x + 4y + 4 = 0\) is
    a focal chord of the parabola
    \(x^2 = -4y\).

13. Show that if \(PQ\) is a focal chord of \(x^2 = 4ay\) where \(P\) is the point
    \((2ap, ap^2)\) and \(Q\) is the point
    \((2aq, aq^2)\) then \(pq = -1\).

14. Find the point of intersection of
    the tangents to the curve
    \(x^2 = 12y\) at \((-6, 3)\) and \((2, \frac{1}{3})\).
15. Show that the tangents to the curve \( x^2 = 4ay \) at \( P(2ap, ap^2) \) and \( Q(2aq, aq^2) \) intersect at the point \([a(p + q), apq]\).

16. (a) Find the equation of the chord joining \( P(-8, 8) \) and \( Q(2, \frac{1}{2}) \) where \( P \) and \( Q \) are points on the parabola \( x^2 = 8y \).
(b) Show that \( PQ \) is a focal chord.

17. Points \( P(2ap, ap^2) \) and \( Q(2aq, aq^2) \) lie on the parabola \( x^2 = 4ay \).
(a) Show that the normal at \( P \) is given by \( x + py = ap^3 + 2ap \).
(b) Find the point \( N \) where this normal meets the axis of the parabola.

18. Point \( M(4, -8) \) lies on the parabola \( x^2 = -2y \).
(a) Find the equation of the focal chord through \( M \).
(b) Find point \( N \) where this chord cuts the parabola again.

19. Tangents are drawn from an external point \( P(-2, -1) \) to the parabola \( x^2 = 12y \).
(a) Find the equation of the chord of contact of the tangents.
(b) Find the coordinates of the points where the tangents meet the parabola.

20. (a) Find the coordinates of the focus \( F \) of the parabola \( x = 12t \), \( y = 6t^2 \).
(b) Find the equation of the focal chord \( PF \) where \( P \) is the point \((6, 1\frac{1}{2})\) on the parabola.
(c) Find \( Q \) where this chord cuts the parabola again.
(d) Find the equations of the tangents to the parabola at \( P \) and \( Q \).
(e) Prove that the tangents are perpendicular.
(f) Find the point of intersection \( R \) of these tangents.
(g) Show that \( R \) lies on the directrix.

21. The chord of contact of two tangents drawn from an external point \( P \) to the parabola \( x^2 = 8y \) has equation \( x + 2y - 3 = 0 \). Find the coordinates of \( P \).

22. Find the equation of the normal to the parabola \( x = 6t, y = 3t^2 \) at the point where \( t = -1 \).

23. Prove that the tangents at the end of a focal chord are perpendicular.

24. Show that the tangents at the ends of a focal chord intersect on the directrix.

25. Show that the equation of the tangent at the point \( P(x_0, y_0) \) on the parabola \( x^2 = 4ay \) is given by \( xx_0 = 2a(y + y_0) \).

Properties of the Parabola

The tangents at the end of a focal chord intersect at right angles on the directrix.
Proof

Let $PQ$ be a focal chord of $x^2 = 4ay$ where $P = (2ap, ap^2)$ and $Q = (2aq, aq^2)$.
Then $pq = -1$
Tangent at $P$ has gradient $m_1 = p$
Tangent at $Q$ has gradient $m_2 = q$

\[ pq = -1 \]

i.e. $m_1m_2 = -1$
\[ \therefore \] the tangents are perpendicular
Tangents intersect at $[a(p + q), apq]$

i.e. $y = apq$
But $pq = -1$
\[ \therefore \] $y = -a$

This is the equation of the directrix.
\[ \therefore \] tangents intersect on the directrix

EXAMPLE

Points $P\left(2, -\frac{1}{2}\right)$ and $Q(-8, -8)$ lie on the parabola $x^2 = -8y$.

(a) Find the equation of line $PQ$.
(b) Show that $PQ$ is a focal chord.
(c) Prove that the tangents at $P$ and $Q$ intersect at right angles on the directrix.

Solution

(a) Equation of $PQ$

\[
\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}
\]

\[
y + 8 = \frac{\frac{1}{2} + 8}{2 + 8}
\]

\[
y + 8 = \frac{3}{4}
\]

\[4y + 32 = 3x + 24\
0 = 3x - 4y - 8\]
(b) \[ x^2 = -8y \]
\[ \therefore 4a = 8 \]
\[ a = 2 \]

Focus = \((0, -2)\)

Substitute \((0, -2)\) into (1)

\[
\text{RHS} = 3(0) - 4(-2) - 8 = 0
\]
\[ = \text{LHS} \]
\[ \therefore PQ \text{ is a focal chord} \]

(c)
\[ y = -\frac{x^2}{8} \]
\[ \frac{dy}{dx} = -\frac{x}{4} \]

At \(P\), \[ \frac{dy}{dx} = -\frac{2}{4} = -\frac{1}{2} \]

The equation formula is
\[ y - y_1 = m(x - x_1) \]
\[ y + \frac{1}{2} = -\frac{1}{2}(x - 2) \]
\[ 2y + 1 = -x + 2 \]
\[ x + 2y = 0 \]  
(1)

At \(Q\), \[ \frac{dy}{dx} = -\frac{(-8)}{4} = 2 \]

The equation formula is
\[ y - y_1 = m(x - x_1) \]
\[ y + 8 = 2(x + 8) \]
\[ = 2x + 16 \]
\[ 0 = 2x - y + 8 \]  
(2)

\(P\) has gradient \(m_1 = -\frac{1}{2}\)

\(Q\) has gradient \(m_2 = 2\)

\[ m_1m_2 = -\frac{1}{2} \times 2 = -1 \]
\[ \therefore \text{the tangents are perpendicular} \]

Solve simultaneous equations to find the point of intersection.

\[
\begin{align*}
x + 2y - 1 &= 0 \\
2x - y + 8 &= 0
\end{align*}
\]
(1)

(1) \times 2: \[ 2x + 4y - 2 = 0 \]  
(3)

(2) - (3): \[ -5y + 10 = 0 \]

CONTINUED
The tangent at point $P$ on a parabola is equally inclined to the axis of the parabola and the focal chord through $P$.

**Proof**

\[ FP = PR \quad \text{(definition of a parabola)} \]

\[ PR = PS + SR \]

\[ = ap^2 + a \]

\[ \therefore \ FP = ap^2 + a \]

Tangent $PQ$ has equation $y - px + ap^2 = 0$
At $Q, x = 0$

$y - 0 + ap^2 = 0$

$y = -ap^2$

∴ $Q = (0, -ap^2)$

$FQ = FO + OQ$

$= a + ap^2$

$= FP$

∴ $\angle FQP = \angle FPQ$ (base $\angle$ s of isosceles $\Delta$)

∴ tangent is equally inclined to the axis and the focal chord.

### Application

This property of the parabola—that is, that the tangent at $P$ is equally inclined to the axis of the parabola and the focal chord through $P$—is used in many practical applications, including telescopes, headlights and radar.

### Class Investigation

Explore the use of the parabola in everyday life. You could go on an excursion to the Observatory, the physics section of a university, an optics manufacturer, an engineering company or a camera manufacturer.

Write about the use of the parabola in any of the above applications, or any others you can think of.
EXAMPLES

1. Find the locus of the midpoints of the chords in the parabola $x^2 = 4ay$ that pass through $(0, 2)$.

Solution

Equation of chord $PQ$ where $P = (2ap, ap^2)$ and $Q = (2aq, aq^2)$ is given by

$$y - \frac{1}{2}(p + q)x + apq = 0.$$  

If $PQ$ passes through $(0, 2)$:

$$2 - \frac{1}{2}(p + q)0 + apq = 0$$

$$apq = -2 \quad (1)$$

For midpoint $M(x, y)$

$$x = \frac{2ap + 2aq}{2} = a(p + q)$$

$$\therefore p + q = \frac{x}{a} \quad (2)$$

$$y = \frac{ap^2 + aq^2}{2} = \frac{1}{2}a(p^2 + q^2) = \frac{1}{2}a[(p + q)^2 - 2pq] \quad (3)$$

Substitute (2) into (3)

$$y = \frac{1}{2}a\left[\left(\frac{x}{a}\right)^2 - 2pq\right]$$

$$2y = \frac{x^2}{a} - 2apq$$

$$= \frac{x^2}{a} + 4 \quad [\text{from (1)}]$$

$$2ay = x^2 + 4a$$

$$x^2 = 2ay - 4a$$

$$= 2a(y - 2)$$

$$\therefore \text{locus is a parabola with vertex (0, 2) and focal length } \frac{a}{2}.$$  

2. Points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$ and chord $PQ$ passes through $(0, -4a)$. Find the locus of the intersection of the normals drawn from $P$ and $Q$. 
Solution

PQ has equation $y - \frac{1}{2}(p + q)x + apq = 0$.

PQ passes through $0, -4a$

\[\therefore - 4a - \frac{1}{2}(p + q)0 + apq = 0\]

\[apq = 4a\]

\[pq = 4\]

(1)

Normals intersect at $[a pq(p + q), a(p^2 + pq + q^2 + 2)]$.

i.e. \[
x = -apq(p + q) = -4a(p + q) \quad \text{[from (1)]}
\]

\[
\frac{x}{-4a} = p + q
\]

\[
y = a(p^2 + pq + q^2 + 2)
\]

\[
= a(p^2 + 4 + q^2 + 2)
\]

\[
= a(p^2 + q^2 + 6)
\]

\[
\frac{y}{a} = p^2 + q^2 + 6
\]

\[
= (p + q)^2 - 2pq + 6
\]

\[
= (p + q)^2 - 8 + 6
\]

\[
= (p + q)^2 - 2
\]

\[
\text{[from (2)]}
\]

\[
\frac{y}{a} + 2 = \frac{x^2}{16a^2}
\]

\[
16ay + 32a^2 = x^2
\]

\[
16a(y + 2a) = x^2
\]

11.10 Exercises

1. (a) Find the equation of the focal chord $PF$ on the parabola $x^2 = 8y$ where $P = (-8, 8)$ and $F$ is the focus.

(b) Find the length of $PF$ where $F$ is the focus.

(c) Show that $PF = FR$ where $R$ is the $y$-intercept of the tangent.

2. (a) Find the equation of the tangent to the parabola $x^2 = 4y$ at the point $P(2p, p^2)$.

(b) Find the length of $PF$ where $F$ is the focus.

(c) Show that $PF = FR$ where $R$ is the $y$-intercept of the tangent.

3. (a) Find the equation of the tangent to the parabola $x^2 = 12y$ at the point $T(6t, 3t^2)$.

(b) Find the coordinates of $Y$, the $y$-intercept of the tangent.

(c) Show that $TF = FY$ where $F$ is the focus.

4. (a) Find the equation of the tangent to the parabola $x^2 = -20y$ at the point $Q(10q, -5q^2)$.
(b) Find the coordinates of \( R \), the \( y \)-intercept of the tangent.
(c) Show that \( \angle FQR = \angle FRQ \) where \( F \) is the focus.

5. (a) Find the equation of chord \( AB \) on the parabola \( x^2 = 12y \) where \( A = \left( 2, \frac{1}{3} \right) \) and \( B = (-18, 27) \).
(b) Show that \( AB \) is a focal chord.
(c) Show that the tangents at \( A \) and \( B \) are perpendicular.
(d) Show that the tangents intersect on the directrix.

6. Find the equation of the locus of the midpoint \( M \) of all chords \( PQ \) where \( P(2ap, ap^2) \) and \( Q(2aq, aq^2) \) lie on the parabola \( x^2 = 4ay \) and \( PQ \) passes through \((0, 2a)\).

7. Find the equations of the tangents to the curve \( x^2 = 8y \) at the points \( P(4p, 2p^2) \) and \( Q(4q, 2q^2) \). Find the equation of the locus of their point of intersection if \( PQ \) is a focal chord.

8. Find the equation of the locus of point \( R \) that is the intersection of the normals at \( P(2p, p^2) \) and \( Q(2a, q^2) \) on the parabola \( x^2 = 4ay \), given that \( pq = -4 \).

9. The chord \( PQ \) is a focal chord of the parabola \( x^2 = 4ay \) where \( P = (2ap, ap^2) \) and \( Q = (2aq, aq^2) \). Find the equation of the locus of the midpoint of \( PQ \).

10. Tangents to the parabola \( x^2 = 4ay \) drawn from points \( P(2ap, ap^2) \) and \( Q(2aq, aq^2) \) intersect at right angles at point \( R \). Find the equation of the locus of
(a) point \( R \)
(b) the midpoint of \( PQ \).

11. The normal at any point \( P(-8p, -4p^2) \) on the parabola \( x^2 = -16y \) cuts the \( y \)-axis at point \( M \). Find the equation of the locus of the midpoint of \( PM \).

12. Given that \( P(2ap, ap^2) \) and \( Q(2aq, aq^2) \) lie on the parabola \( x^2 = 4ay \), chord \( PQ \) subtends a right angle at the origin.

(a) Show \( pq = -4 \).
(b) Find the equation of the locus of the midpoint of \( PQ \).
(c) Show that this locus is a parabola, and find its vertex and focal length.

13. Find the locus of the midpoint of \( PF \) where \( P \) is the point \( (2ap, ap^2) \) on the parabola \( x^2 = 4ay \) and \( F \) is its focus.

14. (a) Find the point of intersection \( T \) of the tangents at \( P(2ap, ap^2) \) and \( Q(2aq, aq^2) \) on the parabola \( x^2 = 4ay \).
(b) Given that \( PQ \) passes through \((0, 6a)\), find the equation of the locus of \( T \).

15. Normals to the parabola \( x = 2at, y = at^2 \) from points \( P(2ap, ap^2) \) and \( Q(2aq, aq^2) \) intersect at \( N \). Find the equation of the locus of \( N \) if \( PQ \) passes through the point \((0, 3a)\).
Class Investigation

Can you spot 6 mistakes in the solution to this question?

Find the equation of the normal to the parabola \( x^2 = 4ay \) at the point \( P(2ap, ap^2) \).

**SOLUTION**

\[
\begin{align*}
x^2 &= 4ay \\
\therefore \quad y &= \frac{x^2}{4a} \\
\frac{dy}{dx} &= -\frac{x}{4a}
\end{align*}
\]

At \( P \), \( \frac{dy}{dx} = \frac{ap^2}{4a} \)

\[
\therefore \quad m_1 = \frac{p^2}{4}
\]

For normal, \( m_1 m_2 = -1 \)

i.e. \( \frac{p^2}{4} m_2 = -1 \)

\[
\therefore \quad m_2 = \frac{4}{p^2}
\]

if \( y - y_1 = m(x - x_1) \)

\[
y - 2ap = \frac{4}{p^2} (x - ap^2)
\]

\[
p^2y - 2ap^2 = 4(x - ap^2)
\]

\[
= 4x - 4ap^2
\]

\[
\therefore \quad p^2y = 4x + 2ap^2
\]
Test Yourself 11

1. Find the equation of the locus of a point moving so that it is equidistant from $A(-1, 2)$ and $B(3, 5)$.

2. Find the equation of the parabola with focus $(2, 1)$ and directrix $y = -3$.

3. Find the radius and centre of the circle $x^2 - 6x + y^2 - 2y - 6 = 0$.

4. Find the coordinates of (a) the vertex and (b) the focus of the parabola $(y + 3)^2 = 12(x - 1)$.

5. (a) Find the coordinates of $P$ on the parabola $x = 4t, y = 2t^2$, where $t = 2$.
   (b) Find the equation of the tangent at $P$.

6. Find the equation of the locus of a point that is always 5 units from the origin.

7. Find (a) the equation of the directrix and (b) the coordinates of the focus of the parabola $x^2 = -8y$.

8. A point $P(x, y)$ moves so that $AP$ and $BP$ are perpendicular, given $A = (3, 2)$ and $B = (-4, 1)$. Find the equation of the locus of $P$.

9. Point $P(x, y)$ is equidistant from the point $A(4, -2)$ and the line $y = 6$. Find the equation of the locus.

10. Find (a) the coordinates of the (i) vertex and (ii) focus and (b) the equation of the directrix of the parabola $x^2 - 2x - 4y + 5 = 0$.

11. Find the equation of the tangent to the parabola $x^2 = 18y$ at the point $(-6, 2)$.

12. Find the length of the diameter of the circle $x^2 + 8x + y^2 - 12y + 3 = 0$.

13. Find the equation of the parabola with directrix $x = 6$ and focus $(-6, 0)$.

14. A parabola has a focus at $(0, 4)$ and its vertex is at $(0, 2)$. Find the equation of the parabola.

15. Find the equation of the locus of a point that is always 3 units from the line $4x - 3y - 1 = 0$.

16. A point is equidistant from the $x$- and $y$-axis. Find the equation of its locus.

17. Find the equation of the parabola with vertex at the origin, axis $y = 0$ and passing through the point $\left(1 \frac{1}{4}, 5\right)$.

18. Find the gradient of (a) the tangent and (b) the normal to the parabola $x^2 = -12y$ at the point where $x = 3$.

19. Find the Cartesian equation of (a) $x = 6t, y = 3t^2$
   (b) $x = -8t^2, y = -16t$.

20. (a) Find the equation of the normal to the parabola $x^2 = 4y$ at the point $(-8, 16)$.
   (b) This normal cuts the parabola again at $Q$. Find the coordinates of $Q$.

21. Show that $7x - 3y + 12 = 0$ is a focal chord of the parabola $x^2 = 16y$.

22. Find the point of intersection of the normals to the parabola $x^2 = -12y$ at the points $\left(4, -1 \frac{1}{3}\right)$ and $\left(-2, -\frac{1}{3}\right)$. 
23. Find the equation of the chord $PQ$ on the parabola $x = 4t, y = 2t^2$ if $t = 5$ at $P$ and $t = -2$ at $Q$.

24. Points $P(10p, 5p^2)$ and $Q(10q, 5q^2)$ lie on the parabola $x^2 = 20y$. Find the equation of the locus of the midpoint of $PQ$ if $pq = -2$.

25. Find the equation of the tangent to the parabola $x = 2at, y = at^2$ the point where $t = 3$.

26. (a) Find the equation of the tangent to the parabola $x^2 = 12y$ at the point $P(6, 3)$. (b) Find $R$, the $y$-intercept of the tangent. (c) Show that $FP = FR$ where $F$ is the focus.

27. (a) Find the equation of the chord $PQ$ given that $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are points on the parabola $x^2 = 4ay$.

(b) If $PQ$ is a focal chord show that $pq = -1$.

28. Find the equation of the normal to the parabola $x = 8t, y = 4t^2$ at the point where $t = -2$.

29. Tangents are drawn from an external point $P(2, -3)$ to the parabola $x^2 = 4y$. (a) Find the equation of the chord of contact of the tangents. (b) Find the coordinates of the points at which each tangent meets the parabola.

30. Chord $PQ$ is a focal chord of $x^2 = 4ay$ where $P = (2ap, ap^2)$ and $Q = (2aq, aq^2)$. Find the equation of the locus of the points of intersection of the tangents at $P$ and $Q$.

### Challenge Exercise 11

1. (a) Find the equation of the locus of point $P$, which is equidistant from fixed points $A(3, 5)$ and $B(-1, 2)$. (b) Show that this locus is the perpendicular bisector of line $AB$.

2. (a) Find the equation of the circle with centre $(1, 3)$ and radius 5 units. (b) Show that the circle cuts the $x$-axis at the points $(5, 0)$ and $(-3, 0)$.

3. Write in Cartesian form the equation $x = \sin \theta, y = \cos 2\theta$.

4. The line with equation $5x - 12y + 36 = 0$ is a chord of the parabola $x^2 = 12y$. Find the point of intersection of the tangents to the parabola from the endpoints of the chord.

5. (a) Find the equation of the normals to the parabola $x^2 = 8y$ at the points $M\left(-2, \frac{1}{2}\right)$ and $N(8, 8)$. (b) Show that these normals are perpendicular. (c) Find the point of intersection $X$ of the normals. (d) Find the equation of line $MN$ and show that it is a focal chord.

6. From which point on the parabola $x^2 = 4ay$ does the normal pass through the focus?

7. (a) Find the equation of the tangents to the parabola $x^2 = 4y$ at the points $A\left(1, \frac{1}{4}\right)$ and $B(-4, 4)$. (b) Show that the point of intersection of these tangents lies on the directrix.
8. Find the equation of the parabola with axis parallel to the $y$-axis and passing through points $(0, -2)$, $(1, 0)$ and $(3, -8)$.

9. Find the equation of the straight line through the centres of the circles with equations $x^2 + 4x + y^2 - 8y - 5 = 0$ and $x^2 - 2x + y^2 + 10y + 10 = 0$.

10. Sketch the region $x^2 + 2x + y^2 - 4y - 4 \leq 0$.

11. (a) Find the equation of the locus of a point $P$ moving so that $PA$ is perpendicular to $PB$ where $A = (-4, 3)$ and $B = (0, 7)$.
   (b) Show that this locus is a circle with centre $(-2, 5)$ and radius $2\sqrt{2}$.

12. Find the exact gradient, with rational denominator, of the normal to the parabola $y^2 = 12x$ at the point where $x = 4$ in the first quadrant.

13. (a) Find the equation of the parabola with vertex $(3, -2)$ and focus $(7, -2)$.
   (b) Find the equation of the tangent to the parabola at the point where $x = 4$ in the first quadrant.

14. Find the exact length of the line from $(2, 7)$ to the centre of the circle $x^2 + 4x + y^2 - 6y - 3 = 0$.

15. Find the equation of the locus of midpoints of all chords of length 2 units in the circle with equation $x^2 + y^2 - 2y - 3 = 0$.

16. A satellite dish is to be 3.5 m wide and 1.1 m deep. Find the position of the focus in millimetres, correct to the nearest millimetre.

17. Find the equation of the locus of point $P$ that moves such that the distance from $P$ to the lines $3x - 4y + 1 = 0$ and $12x + 5y + 3 = 0$ is in the ratio 3:1.

18. $PQ$ is a chord of $x^2 = 4ay$ where $P = (2ap, ap^2)$ and $Q = (2aq, aq^2)$.
   (a) Find the coordinates of point $N$ that divides $PQ$ in the ratio 2:3.
   (b) Find the locus of the midpoint of $PQ$ if $pq = 2$.

19. The chord of contact of the tangents to the parabola $x^2 = 4ay$ from an external point $R(x_1, y_1)$ passes through the point $N(0, 2a)$. Find the equation of the locus of the midpoint of $RN$.

20. (a) Find the coordinates of $T$ where $T$ is the point of intersection of the tangents at the points $t = -2$ and $t = 5$ on the parabola $x = 4t$, $y = 2t^2$.
   (b) Find the coordinates of $P$ where $P$ is the point of intersection of the tangents at the points $X(2at, at^2)$ and $Y(2as, as^2)$ on the parabola $x^2 = 4ay$.
   (c) The tangents from $X$ and $Y$ meet at $45^\circ$. Show that $t = \frac{s - 1}{s + 1}$ or $t = \frac{s + 1}{1 - s}$.